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GEOMETRY AND MENSURATION

236

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GEOMETRY AND MENSURATION

(PART 1)

Serial 1098A
Edition 2

LINES AND PLANE SURFACES

LINES AND ANGLES

DEFINITIONS

1. A **point** indicates position only; it has neither length, breadth, nor thickness.

2. A **line** has only one dimension—length.

3. A **straight line,** Fig. 1, is one that does not change its direction throughout its whole length. A straight line is also frequently called a **right line.**

Fig. 1

4. A **curved line,** Fig. 2, changes its direction at every point.

Fig. 2

5. A **broken line,** Fig. 3, is one made up wholly of straight lines lying in different directions.

Fig. 3

Unless otherwise specified, the word *line* is understood to mean a *straight line.*

6. A **surface** has the shape of a body but is without thickness. A surface has but two dimensions—length and breadth.

7. A **flat surface, plane surface,** or simply a **plane,** is a surface such that a straight line between any two of its

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points lies wholly in the surface. If a straightedge is laid on a plane surface in any direction, every point of the straightedge will touch the surface.

8. A **figure** is any combination of points and lines. A figure that lies entirely in one plane is a **plane figure.** In referring to a figure, a point is designated by a letter placed conveniently near it; thus, in Fig. 1, the left end of the line is referred to as "the point *A*." The entire line is referred to as "the line *A B*," the letters *A* and *B* designating two points, usually the ends of the line. If a line is broken or curved, as many points are named as are considered necessary to designate the line.

9. **Geometry** is that branch of mathematics which treats of the properties of lines, angles, surfaces, and volumes.

10. **Mensuration** is that part of geometry which treats of the measurement of lines, surfaces, and volumes.

ANGLES

11. An **angle,** Fig. 4, is the opening between two straight lines that meet in a point. The two straight lines are the *sides*, and the point where the lines meet is the **vertex** of the angle. Thus, in Fig. 4, the straight lines *O A* and *O B* form an angle at the point *O*; the lines *O A* and *O B* are the sides of this angle, and the point *O* is its vertex.

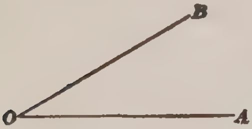


FIG. 4

12. An angle is usually referred to by naming a letter on each of its sides and a third letter at the vertex, the letter at the vertex being placed between the other two. Thus, the angle in Fig. 4 is called angle *A O B* or angle *B O A*.

An angle may also be designated by a letter placed between its sides near the vertex. Thus, the two angles *X C Y* and *Y C Z*, Fig. 5, may be referred to as the angles *A* and *B*, respectively.

An angle that stands alone, that is, an angle whose vertex is not the vertex of any other angle, may be designated by naming the letter at its vertex. For example, the angle in Fig. 4 may be called the angle O.

13. Two angles, as A and B, Fig. 5. having the same vertex and a common side $C\ Y$, are called **adjacent angles.**

14. Any angle may be thought of as being formed, or generated, by a line turning about the vertex as a pivot, from the position of one side to the position of the other. Thus, the angle $A\ O\ B$, Fig. 4, may be conceived as generated by a line turning about O from the position $O\ A$ to the position $O\ B$. The size of the angle does not depend on the length of the sides, which are supposed to be of indefinite length, but on the opening between the sides; or, what is the same thing, on the amount of turning necessary to bring one side to the position of the other.

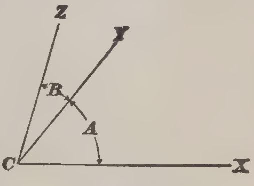


FIG. 5

15. If a straight line, as $A\ B$, Fig. 6, meets another straight line, as $C\ D$, so as to make with it two equal adjacent angles, each of these angles is a **right angle,** and the first line is said to be *normal*, or *perpendicular*, to the second. The point where the first line meets the second is called the *foot* of the perpendicular. It is evident that all right angles are equal.

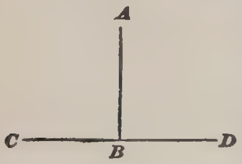


FIG. 6

The edges of the tongue of a carpenter's square, shown in Fig. 7, are perpendicular to those of the body, or long blade, and the angles thus formed are right angles. Thus, $A\ B$ is perpendicular to $B\ C$, and the angle at B is a right angle; $D\ E$ is perpendicular to $E\ F$, and the angle at E is a right angle.

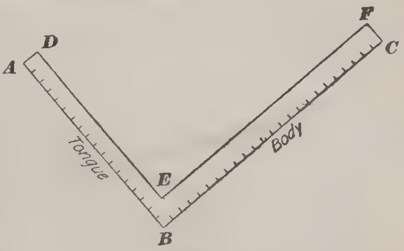


FIG. 7

MEASUREMENT OF ARCS AND ANGLES

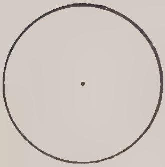


FIG. 8

16. A **circle,** Fig. 8, is a plane figure bounded by a curved line, called the **circumference,** every point of which is equally distant from a point within, called the **center.**

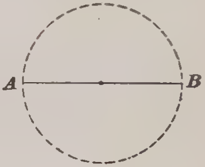


FIG. 9

17. The **diameter** of a circle is a straight line passing through the center and terminated at both ends by the circumference, as $A\,B$, Fig. 9.

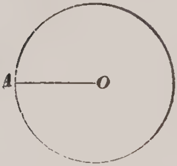


FIG. 10

18. The **radius** of a circle, $O\,A$, Fig. 10, is a straight line drawn from the center to the circumference. It is equal in length to one-half the diameter. The plural of the word radius is *radii.* All radii of any circle are equal in length.

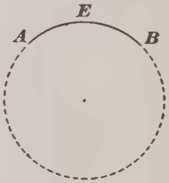
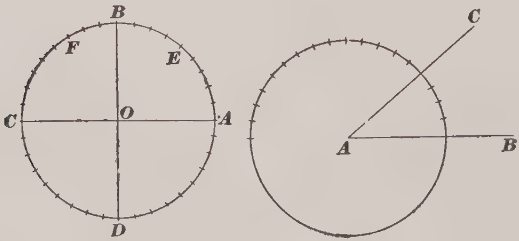


FIG. 11

19. An **arc** of a circle is any part of its circumference, as $A\,E\,B$, Fig. 11.

The whole circumference of every circle is considered as being divided into 360 equal parts, and each of these equal parts is called a **degree;** a degree is divided into 60 equal parts called **minutes;** and a minute is divided into 60 equal parts called **seconds.** Degrees, minutes, and seconds are used as units for measuring circular arcs.



(*a*) FIG. 12 (*b*)

In Fig. 12 (*a*) the circumference is divided into 36 equal parts; hence, each arc between two adjacent division lines contains $360 \div 36 = 10$ degrees. The arc $A\,E\,B$ is one-quarter of a circumference. A quarter of a circumference is called a **quadrant;** its arc contains 360÷4 =90 degrees. The arc *E B F* extends over seven divisions; hence, the arc contains 7×10=70 degrees.

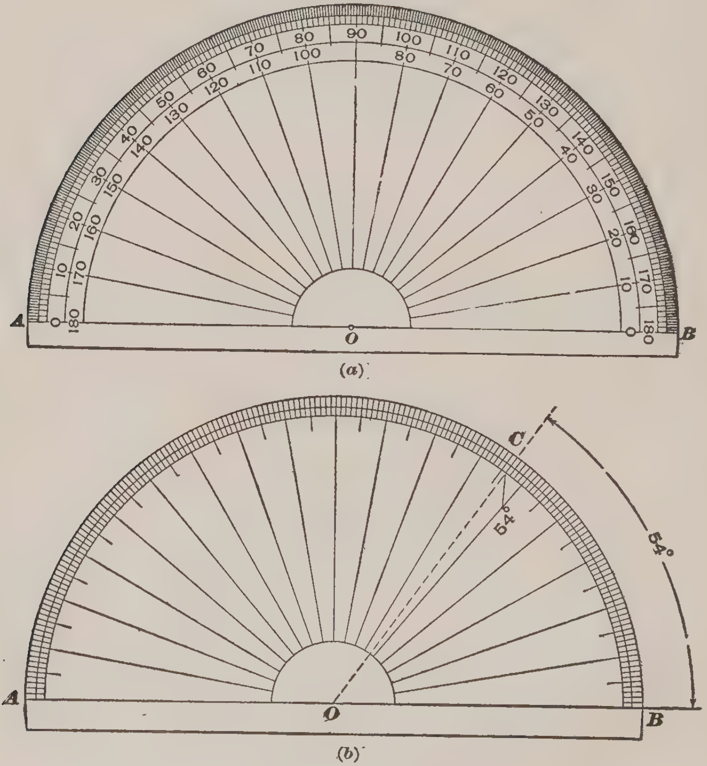
20. Angles, as well as arcs, are measured in degrees, minutes, and seconds. If the vertex of an angle is placed at the center of a circle, the number of degrees in the arc included between the sides of the angle is the number of degrees in the angle. For example, Fig. 12 (*b*) shows a circle of the same size as in (*a*), with half of its circumference divided into short arcs of the same size as in (*a*); that is, each arc between adjacent points of division contains 10 degrees.

Suppose, as indicated in (*b*), that the angle *C A B* is placed with its vertex *A* at the center of the circle. Its sides then include four of the equal small arcs. As each small arc contains 10 degrees, the total arc between *A B* and *A C* contains 4×10=40 degrees; then, the angle *C A B* is said to be an angle of 40°.

In (*a*), the arc *A E B* contains 90 degrees; therefore the angle *A O B* is an angle of 90°. Similarly, the arc *B F C* contains 90 degrees, and so the angle *B O C* is an angle of 90°. This means that the straight line *B O* meets the straight line *C A* in such a way as to form two equal adjacent angles *A O B* and *B O C*; and so, according to Art. **15,** each of these angles is a right angle. In other words, every right angle is an angle of 90°.

21. Angles are commonly measured by the use of a *protractor*, a form of which is shown in Fig. 13 (*a*). It consists of a piece of celluloid or of metal of a semicircular shape and very thin. Along the curved edge are a number of divisions, the smallest representing $\frac{1}{2}$°, or 30′, the next larger representing 1°, and the larger ones representing 5° and 10°, respectively. To use the protractor, it is laid flat on the angle to be measured, with the point *O* directly on the vertex of the angle and the line *A B* directly over one side of the angle. The point where the other side of the angle crosses the scale shows the size of the angle.

22. If a line is given and it is required to draw another line that inclines toward it at a given angle, the required line may be drawn in the following manner: Let *A B*, Fig. 13 (*b*), be the given line and *O* the point from which a line is to be drawn at an angle of 54° with the line *A B*. The protractor, as shown in (*a*), is laid on the line *A B*, as shown in (*b*), so that its center *O* comes directly over the vertex *O* of the angle and so that the line *A B* coincides with the line *A B*. Then 54° is counted off from the lower end *B* of the scale on the protractor. This means 5 large divisions, each of which represents 10°, and 4 small divisions, each of which represents 1°. Opposite the end of the 54° mark the point *C* is located



(*a*)

(*b*)

FIG. 13

with a sharp pencil or a *scriber*, a pointed steel tool used for *scribing*, or scratching, lines on metal surfaces. Then the protractor is removed and a straight line is drawn or scribed so as to pass through C and O. This line CO will make an angle of 54° with the line AB; that is, the angle COB will be 54°.

23. An **oblique angle** is any angle that is not a right angle. An **acute angle** is an oblique angle that is less than a right angle. An **obtuse angle** is an oblique angle that is greater than a right angle. In Fig. 14, BOC and AOC are oblique angles, BOC being an acute angle, and AOC an obtuse angle.

In carpentry, an angle other than a right angle is known as a *bevel angle*, or *bevel;* hence, any acute angle or any obtuse angle is a bevel angle. The instrument used by the carpenter to lay off acute and obtuse angles is called a *bevel*.

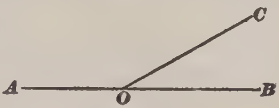


FIG. 14

24. Two angles are said to be **complementary** when their sum is equal to one right angle, or 90°. Each of two complementary angles is called the **complement** of the other. Thus, in Fig. 15, in which AB is perpendicular to BD, the angles M and N are complementary, their sum being equal to the right angle ABD. The angle M is the complement of the angle N; also, the angle N is the complement of the angle M.

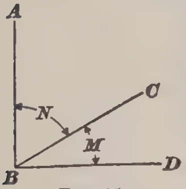


FIG. 15

EXAMPLE.—If the angle CBD, Fig. 15, is 31°, what is the complementary angle ABC?

SOLUTION.—Since ABD is a right angle, the angle ABC is
$90° - \text{angle } CBD = 90° - 31° = 59°$. Ans.

25. Two angles are said to be **supplementary** when their sum is equal to two right angles, or 180°. Each of two supplementary angles is called the **supplement** of the other. In Fig. 16, AOD and DOB are supplementary angles, their sum being evidently equal to the sum of the two right angles POB and

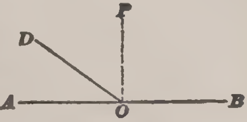


FIG. 16

POA. Thus, AOD is the supplement of DOB, and DOB is the supplement of AOD.

EXAMPLE.—In Fig. 16, the angle AOD is 45°. What is the supplementary angle DOB?

SOLUTION.—Since angle AOD+angle $DOB = 180°$, angle $DOB = 180° -$ angle $AOD = 180° - 45° = 135°$. Ans.

26. The sum of all the angles formed on the same side of a straight line about the same point in the line is equal to two right angles, or 180°. Thus, in Fig. 17, $M+N+P+Q+R=180°$.

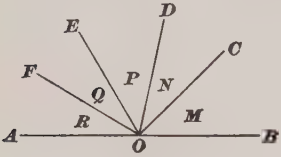


FIG. 17

EXAMPLE.—In Fig. 17, assuming that $R=30°$, $Q=25°$, $P=40°$, and $N=28°$, find M.

SOLUTION.—Since $R+Q+P+N+M=180°$, $M=180°-(R+Q+P+N)=180°-(30+25°+40°+28°)=180°-123°=57°$. Ans.

27. The sum of all the angles formed in the same plane about one point is equal to four right angles. Thus, in Fig. 18,

$$M+N+P+Q+R+S+T+U = \text{four right angles} = 360°.$$

EXAMPLE.—If the angles N, M, U, etc., Fig. 18, are equal to one another, what is the size of each angle?

SOLUTION.—There are eight equal angles and their sum equals 360°. Hence, each angle is

$$360° \div 8 = 45°. \quad \text{Ans.}$$

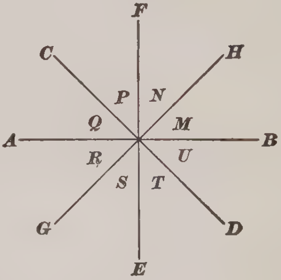


FIG. 18

28. When two lines, as AB and CD, Fig. 19, cut or cross each other, they are said to **intersect.** Their common point O is called their **point of intersection,** or simply their **intersection.**

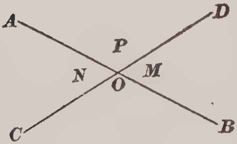


FIG. 19

29. Two intersecting straight lines make four angles having a common vertex. Any one of these angles and the angle on the opposite side of both lines, as the angles M and N, Fig. 19, are called **vertical angles** with respect to each other. Vertical angles may also be defined as those having a common vertex and in which the sides of the one are the prolongations of the sides of the other.

Since M and N are each the supplement of P, they are equal to each other. Any angle is equal to its vertical angle.

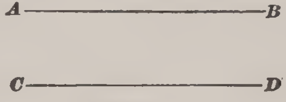


FIG. 20

30. **Parallel lines,** such as $A\,B$ and $C\,D$, Fig. 20, are straight lines that lie in the same plane and never meet, however far they are produced. Any two parallel lines have the same direction and are everywhere equally distant from each other.

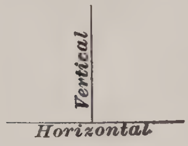


FIG. 21

31. A **horizontal line,** Fig. 21, is a line parallel to the horizon, or to the surface of still water.

32. A **vertical line,** Fig. 21, is a line perpendicular to a horizontal line.

33. In a drawing, it is usual to speak of lines drawn from left to right, or from right to left, as horizontal. In a diagram in a book, a line parallel to the printed lines is called a horizontal line, and a line perpendicular to the horizontal line is said to be vertical.

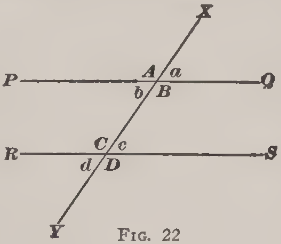


FIG. 22

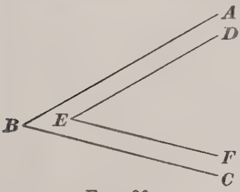


FIG. 23

34. When two parallel lines, as $P\,Q$ and $R\,S$, Fig. 22, are cut by a third line, as $X\,Y$, the angles denoted by the capital letters are equal to one another; that is, $A=B=C=D$. The angles designated by the small letters are also equal to one another: $a=b=c=d$.

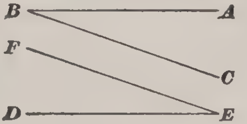


FIG. 24

35. If the corresponding sides of two angles are parallel and lie in the same or in opposite directions, the angles are equal. Thus, if the side $A\,B$, Fig. 23 or Fig. 24, is parallel to the side *D E*, and if the side *B C* is parallel to the side *E F*, the angle *E* is equal to the angle *B*.

36. But if one of the sides of one angle lies in the same direction as the corresponding side of the other angle, and the other two corresponding sides lie in opposite directions, the angles are supplementary. Thus, in Fig. 25, *G H* is parallel to and lies in the same direction as *DE*, and *H I* is parallel to but lies in the opposite direction to *E F*; hence, the angle *G H I* is the supplement of *D E F*.

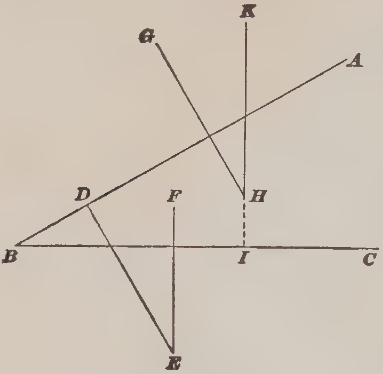


FIG. 25

37. If two sides of an angle are perpendicular to two sides of another angle, the two angles are either equal or supplementary. Thus, if *D E* and *G H*, Fig. 25, are perpendicular to *B A*, and if *E F* and *H K* are perpendicular to *B C*, then angle *E* = angle *B* = angle *G H K*; also, *G H I* is the supplement of *A B C*.

EXAMPLES FOR PRACTICE

1. In Fig. 26, find the number of degrees in the angle *E O A*. Ans. 156°

2. In Fig. 26, find the number of degrees in the arc *A D*. Ans. 113°

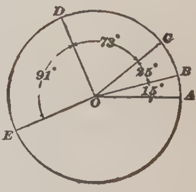


FIG. 26

3. In Fig. 27, find the number of degrees in the angle *B O C*. Ans. 77°

4. In a wheel with fourteen spokes spaced equal distances apart, what is the angle included between the center lines of any two adjacent spokes? Ans. $25\frac{5}{7}$°

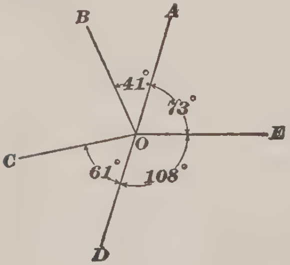


FIG. 27

5. If one straight line, as *O C*, Fig. 14, meets another straight line as *A B*, so as to form an angle *A O C* equal to 137°, what does its adjacent angle *C O B* equal? Ans. 43°

6. If a number of straight lines meet a given straight line at a given point, all being on the same side of the given line, so as to form six equal angles, how many degrees are there in each angle? Ans. 30°

CONVERSION TABLES

38. In making calculations of various kinds, involving the lengths of lines, it may be desirable to find a dimension to the nearest hundredth of a foot, when it is stated in feet and inches; or, a dimension stated as a decimal may have to be expressed as a fraction of an inch. **Conversion tables,** as their name implies, are useful in converting, or changing, inches or fractions of an inch to decimals of a foot or inch, and for changing decimals to fractions. Table I gives the equivalents of inches and fractions of an inch in decimals of a foot, and Table II gives the equivalents of fractions of an inch in decimals of an inch.

TABLE I

INCHES TO DECIMALS OF A FOOT

Inches or Fractions	Decimal of Foot	Approximate Decimal	Inches	Decimal of Foot	Approximate Decimal
$\frac{1}{16}$	.0052	.005	3	.2500	.25
$\frac{1}{8}$	.0104	.010	4	.3333	.33
$\frac{1}{4}$	.0208	.020	5	.4167	.42
$\frac{3}{8}$	.0313	.030	6	.5000	.50
$\frac{1}{2}$	.0417	.040	7	.5833	.58
$\frac{5}{8}$	.0521	.050	8	.6667	.67
$\frac{3}{4}$	.0625	.060	9	.7500	.75
$\frac{7}{8}$	.0729	.070	10	.8333	.83
1	.0833	.080	11	.9167	.92
2	.1667	.170	12	1.0000	1.00

Example 1.—Express 7 feet $8\frac{7}{8}$ inches in feet and a decimal of a foot.

Solution.—From Table I, 8 in. = .67 ft., and $\frac{7}{8}$ in. = .07 ft.; hence, 7 ft. $8\frac{7}{8}$ in. = 7 + .67 + .07 = 7.74 ft. Ans.

EXAMPLE 2.—Express 18.19 feet in feet and inches.

SOLUTION.—By consulting the column of approximate decimals, Table I, it will be observed that .19 ft. is greater than .17 ft., or 2 in., and is less than .25 ft., or 3 in.; .19 being .02 greater than .17, .19 ft. equals 2 in. and .02 ft. more. According to the table, .02 ft. = $\frac{1}{4}$ in.; hence, .19 ft. = $2\frac{1}{4}$ in., and 18.19 ft. = 18 ft. $2\frac{1}{4}$ in. Ans.

TABLE II

FRACTIONS OF AN INCH TO DECIMALS OF AN INCH

Fraction of Inch	Exact Decimal of Inch	Approximate Decimal	Fraction of Inch	Exact Decimal of Inch	Approximate Decimal
$\frac{1}{32}$	.03125	.03	$\frac{9}{16}$	.5625	.56
$\frac{1}{16}$	.06250	.06	$\frac{5}{8}$	.6250	.63
$\frac{1}{8}$	.12500	.13	$\frac{11}{16}$	.6875	.69
$\frac{3}{16}$	.18750	.19	$\frac{3}{4}$	.7500	.75
$\frac{1}{4}$	.25000	.25	$\frac{13}{16}$	.8125	.81
$\frac{5}{16}$	.31250	.31	$\frac{7}{8}$	.8750	.88
$\frac{3}{8}$	.37500	.38	$\frac{15}{16}$	.9375	.94
$\frac{7}{16}$	.43750	.44	1	1.0000	1.00
$\frac{1}{2}$	.50000	.50			

Table II is used in a similar manner. Thus, 9.26 in. = $9\frac{1}{4}$ in., approximately.

PLANE FIGURES

CLASSIFICATION

39. A **plane figure** is any part of a plane surface bounded by straight or curved lines. When a plane figure is bounded by *straight lines*, it is called a **polygon.** The bounding lines are called the *sides*, and the length of the broken line that bounds it (or the whole distance around it) is called the **perimeter** of the polygon.

40. The angles formed by the sides are called the *angles* of the polygon. Thus, *A B C D E*, Fig. 28, is a polygon.

AB, BC, etc. are the sides; EAB, ABC, etc. are the angles: the points A, B, C, D, and E are the *vertexes* of the polygon; and the length of the broken line $ABCDEA$ is the perimeter.

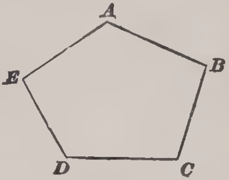


FIG. 28

41. Polygons are classified according to the number of their sides: A polygon of three sides is called a **triangle**; one of four sides, a **quadrilateral**; one of five sides, a **pentagon**; one of six sides, a **hexagon**; one of seven sides, a **heptagon**; one of eight sides, an **octagon**; one of ten sides, a **decagon**; one of twelve sides, a **dodecagon**; etc.

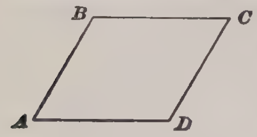


FIG. 29

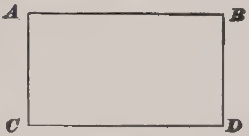


FIG. 30

42. **Equilateral polygons** are those in which the sides are all equal. Thus, in Fig. 29, $AB=BC=CD=DA$; hence, $ABDC$ is an equilateral polygon.

43. An **equiangular polygon** is a polygon whose angles are all equal. Thus, in Fig. 30, angle A = angle B = angle D = angle C; hence, $ABCD$ is an equiangular polygon.

44. A **regular polygon** is a polygon in which all the sides and all the angles are equal. Thus, in Fig. 31, $AB=BD=DC=CA$; and angle A = angle B = angle D = angle C; hence, $ABDC$ is a regular polygon. A regular polygon having four sides is called a *square*. Some other regular polygons, with their names, are shown in Fig. 32.

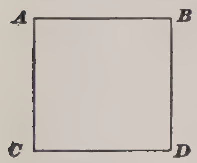
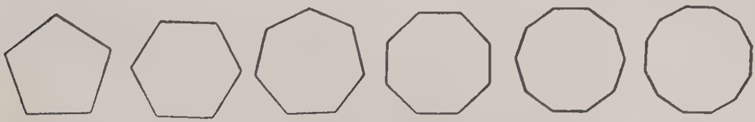


FIG. 31



Pentagon Hexagon Heptagon Octagon Decagon Dodecagon

FIG. 32

TRIANGLES

45. According to the relative lengths of their sides, triangles are said to be *isosceles*, *equilateral*, or *scalene;* and according to the relative sizes of their angles they are said to be *right triangles* or *oblique triangles*.

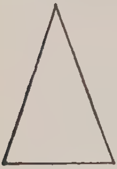


FIG. 33

46. An **isosceles triangle,** Fig. 33, is one having two of its sides equal.

47. An **equilateral triangle,** Fig. 34, is one that has its three sides equal.

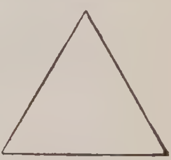


FIG. 34

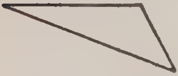


FIG. 35

48. A **scalene triangle,** Fig. 35, is one having no two of its sides equal.

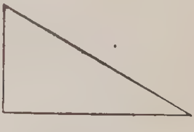


FIG. 36

49. A **right-angled triangle,** Fig. 36, is any triangle having one right angle. The side opposite the right angle is called the **hypotenuse;** when the *sides* of a right-angled triangle are referred to, the short sides are usually meant. For brevity, a right-angled triangle is termed a **right triangle.**

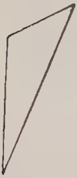


FIG. 37

50. An **oblique-angled,** or **oblique, triangle**, Fig. 37, is one that has no right angle. Thus, the isosceles, equilateral, and scalene triangles shown in Figs. 33 to 35 are also oblique triangles.

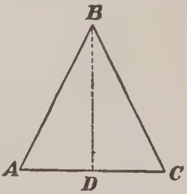


FIG. 38

51. The **base** of any triangle is the side on which the triangle is supposed to stand. A triangle may be drawn so as to stand on any side as the base.

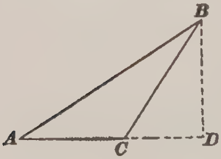


FIG. 39

The **altitude** of any triangle is a line drawn from the vertex of the angle opposite the base perpendicular to the base, or to the base extended. Thus, in Figs. 38 and 39, if the side *A C* is the base of the triangle, the line *B D* is the altitude, which, in Fig. 39, is drawn to the base *A C* extended to *D*. In a right triangle, as in Fig. 36, if one of the short sides is taken as the base, the other short side will be the altitude of the triangle.

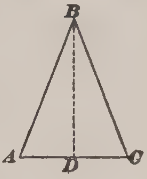


Fig. 40

52. In an isosceles triangle, the angles opposite the equal sides are equal. Thus, in Fig. 40, $AB = BC$; hence, angle $C =$ angle A.

In any isosceles triangle, if a perpendicular is drawn from the vertex opposite the unequal side to that side, it bisects (cuts in halves) the side. Thus, in the isosceles triangle *A B C*, Fig. 40, the line *B D*, drawn from the vertex *B* opposite the unequal side *A C* perpendicular to that side, divides it into two equal parts *A D* and *D C*.

Similarly, in an isosceles triangle, if a line is drawn from the middle point of the unequal side to the vertex of the angle opposite, it will divide that angle into two equal parts, or bisect it. Thus, if *D* is the middle point of *A C* in the isosceles triangle *A B C*, and *D B* is drawn, the angle *A B D* will equal the angle *D B C* and each will be half of the angle *A B C*. The line *D B* divides the triangle into two equal right triangles *A D B* and *C D B*.

If two angles of any triangle are equal, the triangle is isosceles, and the sides opposite the equal angles are of equal lengths.

53. In any triangle, the sum of the three angles is equal to two right angles, or 180°. Thus, in Fig. 41, the sum of the angles at *A*, *B*, and $C =$ two right angles; that is, $A + B + C = 180°$. Hence, if any two angles of a triangle are given, the third may be found by subtracting the sum of the two from 180°.

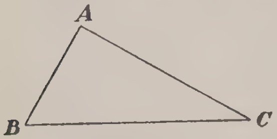


Fig. 41

Example.—If, in Fig. 41, $A = 90°$ and $C = 35°$, how many degrees are there in *B*?

Solution.—Since the sum of *A*, *B*, and *C* is 180°, *B* is equal to the difference between 180° and the sum of *A* and *C*. Therefore,

$$B = 180° - (90° + 35°) = 180° - 125° = 55°. \quad \text{Ans.}$$

54. In any right triangle there can be but one right angle; and since the sum of all the angles equals two right angles, it is evident that the sum of the two acute angles must be equal to a right angle. Therefore, if in any right triangle one acute angle is known, the other can be found by subtracting the known angle from 90°. Thus, $A\,B\,C$, Fig. 42, is a right triangle, right angled at C. Then, angle $A+$ angle $B=$one right angle$=90°$. If $A=40°30'$, $B=90°-40°30'=49°30'$.

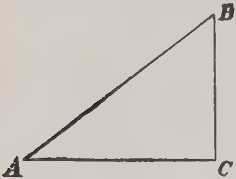


FIG. 42

If an isosceles triangle has one angle a right angle, each of the other two angles is 45°. Thus, if $A\,C$ and $B\,C$ were equal, the angles A and B would each be 45°.

55. Isosceles triangles and right triangles are met with very frequently in carpentry and in other forms of building construction. For example, Fig. 43 shows the outline of a common gable on a house. The lines $A\,B$ and $B\,C$ represent the rafters that support the roof and $B\,D$ represents the height of the gable above the base $A\,C$. As the rafters are of the same length on both sides of the roof, $A\,B=B\,C$, and so the triangle $A\,B\,C$ representing the gable is an isosceles triangle.

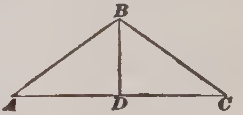


FIG. 43

The line $B\,D$, representing the height of the gable, is a perpendicular from B to $A\,C$. According to the statements made in Art. **52**, therefore, it bisects $A\,C$ and divides the triangle $A\,B\,C$ into two equal right triangles $A\,D\,B$ and $C\,D\,B$. In other words, $A\,D=D\,C$, and the angle $A\,B\,D$ is equal to the angle $C\,B\,D$. As $A\,B\,C$ is an isosceles triangle, the angles A and C are equal.

56. Carpenters give special names to the various dimensions of a gable. For instance, the height $B\,D$, Fig. 43, is called the *rise*. The distance $A\,D$ from the point A to the foot of a plumb-line dropped from B, or to D, is called the *run*. As $C\,D=A\,D$, the run is also represented by $C\,D$. The total width $A\,C$, which is twice the run, is called the *span*.

The slope of the roof depends on the relative height *B D* and width *A C*. The greater the rise, or height, for a given span, or width, the steeper becomes the slope, or *pitch*, of the roof. The pitch is the ratio of the rise to the span; thus, if the rise *B D* is 8 feet and the span, or width, *A C* is 24 feet, the pitch is $\frac{8}{24}=\frac{1}{3}$.

57. Another example of the occurrence of triangles in building work is found in the roof truss, which is a frame built to support a roof. The outline of a simple form of truss is shown in Fig. 44. The distance *A C* represents the span of the roof from wall to wall of the building, *B A* and *B C* are the parts corresponding to the rafters, and *B D* is the rise, or height, of the roof. The lines *D E* and *D F* indicate braces that stiffen and strengthen the truss.

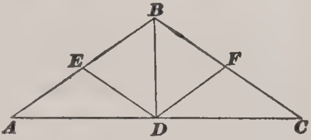


FIG. 44

EXAMPLES FOR PRACTICE

1. Find the number of degrees in each of the angles *B* and *C* of the triangle *A B C*, Fig. 45. Ans. $\begin{cases} B=36° \\ C=108° \end{cases}$

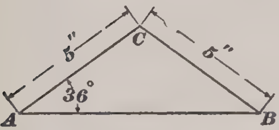


FIG. 45

2. How many degrees are there in each of the acute angles of an isosceles right triangle? Ans. 45°

3. One of the acute angles of a right triangle contains 30°. How many degrees are there in the other acute angle? Ans. 60°

4. In the roof truss, shown in Fig. 44, angle $A=35°$ and $A E=E B=E D$. Find the number of degrees in the angles *A E D*, *E D A*, and *A B D*. Ans. $\begin{cases} A E D=110° \\ E D A=35° \\ A B D=55° \end{cases}$

5. In a gable like that shown in Fig. 43, the span *A C* is 18 feet and the rise *B D* is 6 feet. What is the pitch, or slope? Ans. $\frac{1}{3}$

PRINCIPLE OF RIGHT TRIANGLE

58. A very important principle, and one that is applied in the solution of a great variety of problems, is the following: In every right triangle, the square formed on the hypotenuse is equal to the sum of the squares formed on the other two sides. This is clearly shown in Fig. 46, in which $A\,B\,C$ is a right triangle with its right angle at B. For convenience, the sides of the triangle are taken to represent 3, 4, and 5 feet, as indicated by the figures. The hypotenuse $A\,C$ forms one side of a square $A\,C\,E\,D$; the side $A\,B$ forms one side of a square $A\,B\,I\,K$; and the side $B\,C$ forms one side of a square $B\,C\,H\,F$. By dividing each of these squares into smaller squares of equal sizes, it will be seen that there are 25 small squares in $A\,C\,E\,D$, 16 in $B\,C\,H\,F$, and 9 in $A\,B\,I\,K$; in other words, the square $ACED$ on the hypotenuse contains as many smaller squares as the squares $A\,B\,I\,K$ and $B\,C\,H\,F$ together, or, $25=9+16$. This principle may be stated in another way by saying that the square of the length of the hypotenuse is equal to the sum of the squares of the other two sides. In the illustration, the square of the short side is $3^2=9$ and of the long side is $4^2=16$, and the square of the hypotenuse is $5^2=25$, which is equal to $9+16$; hence, $3^2+4^2=5^2$.

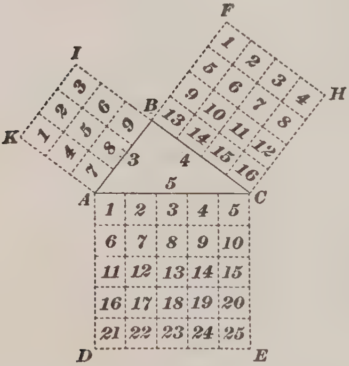


FIG. 46

According to this principle, if the lengths of the sides $A\,B$ and $B\,C$ are known, the length of the hypotenuse can be found by adding the squares of the lengths of the sides $A\,B$ and $B\,C$, and then extracting the square root of the sum.

Rule.—*To find the hypotenuse of a right triangle, add together the squares of the sides and extract the square root of the sum.*

Stated as a formula, this rule becomes

$$c=\sqrt{a^2+b^2}$$

in which c = length of hypotenuse;
a = length of one short side;
b = length of other side.

EXAMPLE 1.—In Fig. 47, $B\ C$ is a brace joining the two pieces $A\ B$ and $A\ C$, which are perpendicular to each other. Find the length of $B\ C$, B being 24 inches from A, and C 36 inches from A.

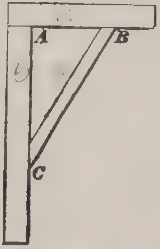


FIG. 47

SOLUTION.—$A\ B\ C$ is a right triangle, of which $A\ B$ and $A\ C$ are the sides and $B\ C$ the hypotenuse. The length of $B\ C$ is found by adding the squares of the lengths of $A\ B$ and $A\ C$ and extracting the square root of the sum. Thus, the square of $A\ B$ equals

$$24^2=24\times24=576$$

The square of $A\ C$ is

$$36^2=36\times36=1{,}296$$

The sum of these squares is $576+1{,}296=1{,}872$; the square root of $1{,}872=43.27$. The length of $B\ C$ is therefore 43.27 in., or $43\frac{1}{4}$ in., approximately. Ans.

EXAMPLE 2.—In a gable like that shown in Fig. 43, the span $A\ C$ is 20 feet and the height $B\ D$ is 10 feet. What is the length of the rafter $A\ B$?

SOLUTION.—Since D is the middle point of $A\ C$, $A\ D=\frac{1}{2}\ A\ C=10$ ft. By Art. 52, $A\ D\ B$ is a right triangle. Therefore, the square of the hypotenuse $A\ B$ is equal to the sum of the squares of the sides $A\ D$ and $B\ D$. As $A\ D=10$ ft. and $B\ D=10$ ft., $\overline{A\ D}^2=100$ and $\overline{B\ D}^2=100$. Then, $\overline{A\ B}^2=100+100=200$, and $A\ B=\sqrt{200}=14.14$ ft., or 14 ft. $1\frac{11}{16}$ in. Ans.

59. When one side and the hypotenuse of a right triangle are known, the other side may be found by applying the following rule:

Rule.—*To find one side of a right triangle when the other side and the hypotenuse are known, subtract the square of the given side from the square of the hypotenuse and extract the square root of the difference.*

Using letters having the same meaning as those given in the formula of Art. 58, this rule may be stated thus:

$$a=\sqrt{c^2-b^2} \text{ or } b=\sqrt{c^2-a^2}$$

EXAMPLE 1.—It is desired to find the height of the ceiling in a room. When one end of a 10-foot pole is set in the angle of the ceiling and wall, the other end is 6 feet from the base of the wall. What is the height of the ceiling?

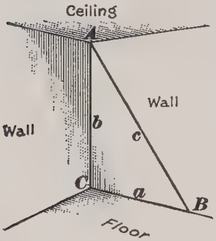


FIG. 48

SOLUTION.—As the wall is at right angles to the ceiling and to the floor, the 10-ft. pole occupies the position AB, Fig. 48, with its lower end B at a distance of 6 ft. from C. As the line AC where the walls meet is perpendicular to the floor, ACB is a right triangle with its right angle at C, and so AB is the hypotenuse. Call the hypotenuse c and the sides a and b, as shown. Then, according to the formula just given, $b=\sqrt{c^2-a^2}$. But, $c^2=10^2=100$, and $a^2=6^2=36$. Then,

$$b=\sqrt{100-36}=\sqrt{64}=8 \text{ ft.}$$

Therefore, the height of the ceiling is 8 ft. Ans.

EXAMPLE 2.—The length of each of the equal sides of an isosceles triangle is 10 inches, and the length of the base is 12 inches; find the length of the perpendicular from the vertex to the base.

SOLUTION.—Let ABC, Fig. 49, be the isosceles triangle. Then, AD is one-half of the base; therefore, $AD=6$ in. In the right triangle ACD, the side CD is equal to the square root of the remainder obtained by subtracting the square of the side AD from the square of the hypotenuse AC. Hence,

$$CD=\sqrt{10^2-6^2}=\sqrt{64}=8 \text{ in. Ans.}$$

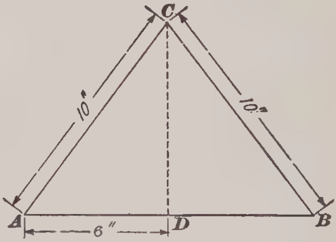


FIG. 49

60. The hypotenuse of a right triangle whose sides are of equal length is equal to the square root of twice the square of a side. ABC, Fig. 50, is a right triangle, the side AB being equal to the side BC; then,

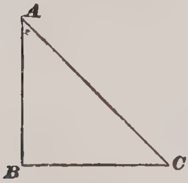


FIG. 50

$$AC=\sqrt{2\times\overline{AB}^2}=1.4142\ AB$$

Rule.—*The hypotenuse of a right triangle having equal sides is equal to 1.4142 times the length of a side.*

NOTE.—In practice, when great accuracy is not required, $1\frac{5}{12}$ is sometimes used instead of 1.4142, as given in the foregoing rule. For example, when each side is 12 inches the hypotenuse is practically $1\frac{5}{12}\times 12=17$ inches.

EXAMPLE.—What is the length of a rafter, if the rise is 15 feet, and the span is 30 feet?

SOLUTION.—Since the rise DG, Fig. 51, is equal to one-half of the span, $DG = \frac{1}{2}EF = EG$. The length ED of the rafter is therefore the hypotenuse of a right triangle whose sides are equal.

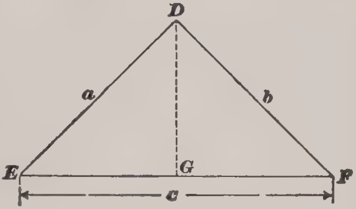


FIG. 51

$EG = \frac{1}{2}$ of $30 = 15$ ft.
$ED = 1.4142 \times 15 = 21.21$ ft.
$.21$ ft. $= .21 \times 12 = 2.52$ in. $= 2\frac{1}{2}$ in., nearly

Therefore, the length of the rafter is 21 ft. $2\frac{1}{2}$ in. Ans.

The diagonal of a square is the hypotenuse of a right triangle having equal sides; hence, the diagonal of a square is equal to 1.4142 times the side of the square. The side of a square can be obtained when the diagonal is known by dividing the diagonal by 1.4142, or by multiplying the diagonal by .7071.

61. A right angle can be laid off by the following method, when the three sides of any right triangle are known: Suppose that AB, Fig. 52, is to be one line of the foundation of a building and it is desired to lay off from A another line that will make a right angle with AB.

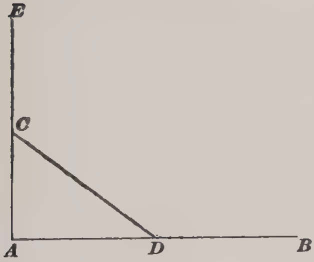


FIG. 52

It is known that a triangle whose sides are respectively 6, 8, and 10 feet is a right triangle, since $6^2 + 8^2 = 10^2$. On AB lay off AD equal to 8 feet. Fasten one end of a tape, or line, at A and draw the other end toward E, so that AE will be as nearly perpendicular to AB as can be estimated by the eye. Make AC on AE 6 feet. Measure the distance CD; if this is more or less than 10 feet, swing the line AE to the right or left, as may be necessary, until DC is exactly 10 feet, when the angle at A will be a right angle.

Any distances that will form three sides of a right triangle may be used in the preceding method; the numbers 3, 4, and 5, or some multiple, as 6, 8, and 10, or 9, 12, and 15, are most convenient.

62. The principle of the right triangle is of very great value in practical work, and the student should become thoroughly familiar with it in all of its variations.

The following is an example showing a double application of the right-triangle principle:

EXAMPLE.—The outline in Fig. 53 represents a skylight with a four-sided roof, each side in the form of an isosceles triangle. The length of the base $A D$ is 9 feet and the width $C D$ is 7 feet 6 inches. The point O is 2 feet above the plane of $A B C D$. What is the length of the edge $O D$ at the angle of the skylight?

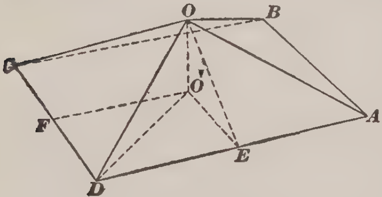


FIG. 53

SOLUTION.—It will be seen that the point O' is directly under O and 2 ft. below it, so that O' is on the same level as $A B C D$. It will also be seen that $D O$ is simply the hypotenuse of a right triangle, whose base is $D O'$ and whose altitude is $O O'$. But $D O'$ is also the hypotenuse of a right triangle, whose sides are $E O'$ and $E D$. $E O'$ is one-half of 7 ft. 6 in., or 3.75 ft., and $E D$ is one-half of 9 ft., or 4.5 ft. Then,

$$(D O')^2 = 3.75^2 + 4.5^2 = 34.3125$$

$$D O' = \sqrt{34.3125} = 5.86 \text{ ft.}$$

$$(D O)^2 = (D O')^2 + (O O')^2 = 34.3125 + 4 = 38.3125$$

$$D O = \sqrt{38.3125} = 6.19 \text{ ft.} = 6 \text{ ft. } 2\tfrac{1}{4} \text{ in., nearly.} \quad \text{Ans.}$$

SIMILAR FIGURES

63. Similar polygons are those whose corresponding angles are equal and whose corresponding sides are proportional. In the quadrilaterals $A B C D$ and $A'' B'' C'' D''$, Fig. 54, the corresponding angles are equal; that is, angle A equals angle A''; angle B equals angle B''; angle C equals angle C''; and angle D equals angle D''. Moreover, the corresponding sides of the figures are proportional; that is, $A'' B''$ bears the same ratio to $A B$ as does $B'' C''$ to $B C$, $C'' D''$ to $C D$, and $D'' A''$ to $D A$. The quadrilaterals $A B C D$ and $A'' B'' C'' D''$ are therefore similar.

The quadrilaterals $A B C D$ and $A' B' C' D'$ have their corresponding angles equal, but the figures are not similar, since the corresponding sides are not proportional. For example, $A' D'$ does not bear the same ratio to $A D$ as does $A' B'$ to $A B$.

64. Two triangles are similar when the angles of one are equal to the angles of the other. In the triangles $A B C$ and $A' B' C'$, Fig. 55, angle A equals A', angle B equals B', and angle C equals angle C'; the triangles are therefore similar, and their corresponding sides are proportional.

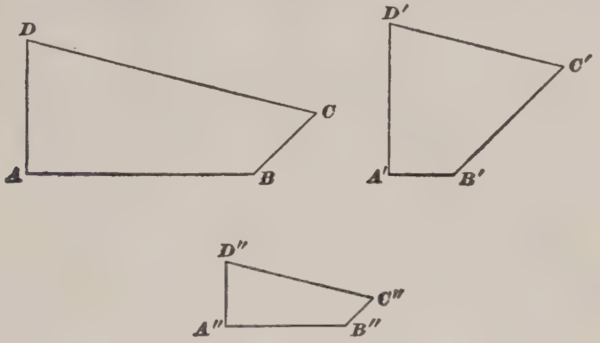


FIG. 54

65. Triangles that have their corresponding sides parallel are similar; for, the sides of the triangles are the sides of the angles, and it was shown in Art. **35** that if the sides of two angles are parallel, the angles are equal. In Fig. 55, $A B$ is parallel to $A' B'$, and $B C$ to $B' C'$, and $A C$ to $A' C'$; therefore, the angles A, B, and C are equal to A', B', and C', respectively, and the triangles are similar.

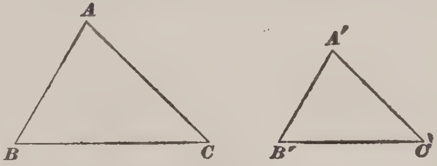


FIG 55

The following are some of the proportions that may be formed from these similar triangles:

$$A B : A' B' = A C : A' C'$$
$$A B : A' B' = B C : B' C'$$
$$A C : A' C' = B C : B' C'$$
$$A B : B C = A' B' : B' C'$$

EXAMPLE.—In Fig. 55, $A\,B=6$ feet, $A\,C=7$ feet, $B\,C=8$ feet, and $B'\,C'=6$ feet; find $A'\,B'$ and $A'\,C'$.

SOLUTION.—It will be observed that the lengths of the similar sides $B\,C$ and $B'\,C'$ are given, the ratio between their lengths being 8:6, or $\frac{8}{6}$. The ratio between other similar sides will be the same; that is, $\frac{A\,B}{A'\,B'}=\frac{8}{6}$, and $\frac{A\,C}{A'\,C'}=\frac{8}{6}$. Since

$$A\,B : A'\,B'=8:6, \text{ or } 6:A'\,B'=8:6$$

$$A'\,B'=\frac{6\times 6}{8}=4\tfrac{1}{2}\text{ ft.}=4\text{ ft. }6\text{ in.}\quad\text{Ans.}$$

Again, $A\,C : A'\,C'=B\,C : B'\,C'$, or $7 : A'\,C'=8:6$

Then, $$A'\,C'=\frac{7\times 6}{8}=5\tfrac{1}{4}\text{ ft.}=5\text{ ft. }3\text{ in.}\quad\text{Ans.}$$

66. If a straight line is drawn through two sides of a triangle parallel to the third side, it divides those sides proportionally. Thus, let the line $D\,E$ be drawn parallel to the side $B\,C$ in the triangle $A\,B\,C$, Fig. 56. Then, $A\,D : A\,B=A\,E : A\,C$, or $A\,D : D\,B=A\,E : E\,C$. It is to be noticed that the triangles $A\,D\,E$ and $A\,B\,C$ are similar, and their sides are proportional. Thus $A\,B : A\,D=B\,C : D\,E$, and $A\,C : A\,E=B\,C : D\,E$.

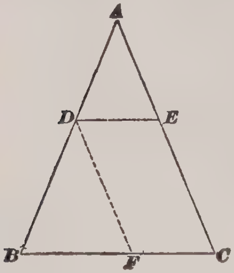


FIG. 56

If a straight line, as $D\,F$, is drawn from D, parallel to $A\,C$, the triangles $A\,D\,E$, $A\,B\,C$, and $D\,B\,F$ are all similar, and a number of other proportions may be formed.

EXAMPLE 1.—In Fig. 57, $D\,E$ is parallel to $B\,C$, and $A\,D$ is one-third of $A\,B$; find the length of $A\,E$.

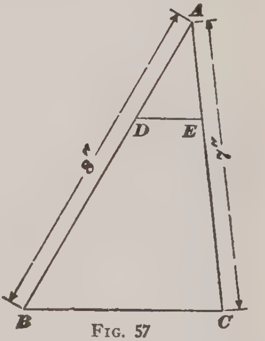


FIG. 57

SOLUTION.—Since $D\,E$ is parallel to $B\,C$, $A\,D : A\,B=A\,E : A\,C$. But, $A\,D : A\,B=1:3$; then, $A\,E : A\,C=1:3$. Substituting for $A\,C$ its value, $A\,E : 7=1:3$; hence

$$A\,E=\frac{7\times 1}{3}=2\tfrac{1}{3}\text{ in.}\quad\text{Ans}$$

Example 2.—On a drawing, it is required to divide a line 8 inches long into twelve parts.

Solution.—The principle of similar triangles is very useful in solving an example of this kind. Let $A\,B$, Fig. 58, be the line 8 in. long that is to be divided into 12 equal parts. Obviously, each part will be $8 \div 12 = \frac{2}{3}$ in.; but the usual scales and rules are not graduated in thirds, and so they cannot be used. So, from one end A of $A\,B$ draw a line $A\,C$ at an acute angle to $A\,B$. The size of the angle is not important. From A to C lay off twelve equal distances, as, for example, twelve equal spaces of 1 in. each, which can be done readily with a scale or ruler. Join C and B by a straight line $C\,B$. Then, from the points of division on $A\,C$, draw lines parallel to $B\,C$, and these lines will divide $A\,B$ into twelve equal parts of $\frac{2}{3}$ in. each.

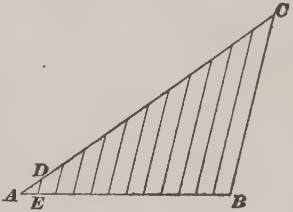


FIG. 58

The proof is quite simple. The lines parallel to $B\,C$ form a series of triangles similar to $A\,B\,C$, and their corresponding sides are therefore proportional. For example, the triangle $A\,E\,D$ is similar to $A\,B\,C$, and so $A\,E : A\,D = A\,B : A\,C$. But, $A\,D = 1$ in., $A\,B = 8$ in., and $A\,C = 12$ in. Hence, substituting these values, $A\,E : 1 = 8 : 12$, or $A\,E = \frac{1 \times 8}{12} = \frac{2}{3}$ in.

The equal spaces on $A\,C$ may be any convenient length instead of 1 in.; for example, they may be $\frac{1}{2}$ in. each, or $\frac{3}{4}$ in. or $\frac{7}{8}$ in., or any other length. The result will be the same in any case.

EXAMPLES FOR PRACTICE

1. A ladder 65 feet long reaches to the top of a house when its foot is 25 feet from the house. How high is the house, supposing the ground to be level? Ans. 60 ft.

2. In a triangle $A\,B\,C$, side $A\,B = 32$ feet, $B\,C = 34$ feet, and $A\,C = 48$ feet. If side $A\,B$ of a similar triangle is 72 feet long, what are the lengths of the other two sides? Ans. $\begin{cases} A\,C = 108 \text{ ft.} \\ B\,C = 76 \text{ ft. } 6 \text{ in.} \end{cases}$

3. The base of a right triangle is 24 inches, and its altitude, 72 inches. At what distance from the top is the triangle 16 inches wide? Ans. 48 in.

4. What length of rafter is required for each side of a gable, if the span is 24 feet, and the rise is three-fourths the span? Ans. 21 ft. $7\frac{1}{2}$ in.

5. $A\,B\,C\,D$, Fig. 59, is the outline of a roof. It is desired to extend the sides $A\,B$ and $C\,D$ to the ridge E. What are the distances $E\,F$ and $E\,A$?

Ans. $\begin{cases} E\,F = 7 \text{ ft. } 6 \text{ in.} \\ E\,A = 22.41 \text{ ft., or } 22 \text{ ft. } 5 \text{ in., nearly} \end{cases}$

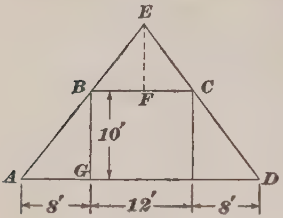


Fig. 59

6. What is the distance between the 16-inch mark on the body of a steel square, Fig. 7, and the 12-inch mark on the tongue?

Ans. 20 in.

7. The distance $E\,B$, Fig. 60, is 8⅓ feet. If the distance between the pieces parallel to $A\,B$, measured from center to center, is 20 inches, and the length of $A\,B$ is 5 feet 6 inches, what is the length of the piece $C\,D$?

Ans. 3 ft. 3⅗ in.

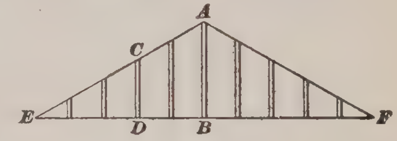


Fig. 60

GEOMETRICAL PRINCIPLES OF THE CIRCLE

67. A **chord** is a straight line joining any two points in a circumference; or, it is a straight line joining the extremities of an arc. Thus, in Fig. 61, $A\,B$ is the chord of the arc $A\,E\,B$.

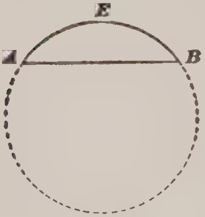


Fig. 61

68. A **segment** of a circle is the space included between an arc and its chord. Thus, in Fig. 61, the space between the arc $A\,E\,B$ and the chord $A\,B$ is a segment.

69. A **sector** of a circle is the space included between an arc and two radii drawn to the extremities of the arc. Thus, in Fig. 62, the space included between the arc $A\,B$ and the radii $O\,A$ and $O\,B$ is a sector of the circle.

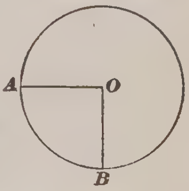


Fig. 62

70. Two circles are equal when the radius or diameter of one is respectively equal to the radius or diameter of the other.

Two arcs are equal when the radius and chord of one are equal to the radius and chord of the other.

71. An **inscribed angle** is one whose vertex lies on the circumference of a circle, and whose sides are chords. It is measured by one-half the intercepted arc. Thus, in Fig. 63, *A B C* is an inscribed angle, and it is measured by one-half the arc *A D C*.

Example.—If, in Fig. 63, the arc *A D C* contains 144°, what is the measure of the inscribed angle *A B C*?

Solution.—Since the angle is an inscribed angle, it is measured by one-half the intercepted arc, or

$\frac{1}{2} \times 144° = 72°$. Ans.

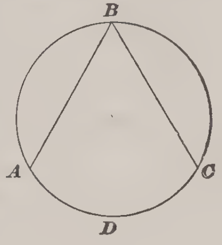


Fig. 63

72. All angles inscribed in the same segment are equal. Thus, in Fig. 64, the inscribed angles *A B E*, *A C E*, and *A D E* are all equal; for each is measured by one-half of the arc *A E*.

For example, if the arc *A E* is 86°, each of the angles *A B E*, *A C E*, and *A D E* is equal to one-half of 86°, or 43°. It is thus seen that the angles *A B E*, *A C E*, and *A D E* are each equal to one-half of the angle *A O E*, which is formed by drawing radii from the center of the circle *O* to *A* and *E*.

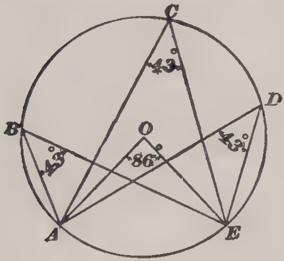


Fig. 64

Any angle inscribed in a semicircle is a right angle. Thus, the angle *A C B*, Fig. 65, is a right angle, for it is measured by one-half of the semicircumference *A D B*, and one-half of a semi-circumference is a quadrant, or 90°.

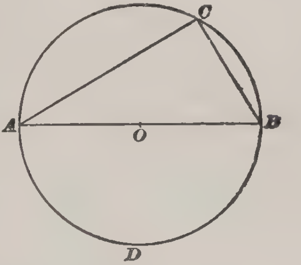


Fig. 65

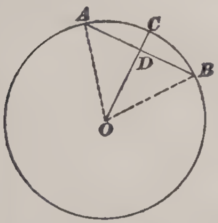


Fig. 66

73. If, in any circle, a radius is drawn perpendicular to any chord, it will bisect the chord. Thus, if the radius *O C*, Fig. 66, is perpendicular to the chord *A B*, then *A D* = *D B*.

A radius that bisects a chord, bisects also the angle included between radii drawn to the ends of the chord.

If a straight line is drawn perpendicular to any chord at its middle point, it must pass through the center of the circle.

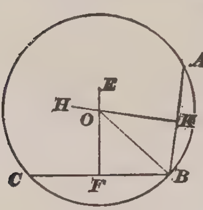


FIG. 67

74. Through any three points not in the same straight line, a circumference can be drawn. Let A, B, and C, Fig. 67, be any three points. Join A and B, also B and C, by straight lines. From the middle point of $A\,B$ draw $K\,H$ perpendicular to $A\,B$; from the middle point of $B\,C$ draw $F\,E$ perpendicular to $B\,C$. These two perpendiculars intersect at O. With O as a center, and $O\,B$, $O\,A$, or $O\,C$ as a radius, describe a circle: it will pass through A, B, and C.

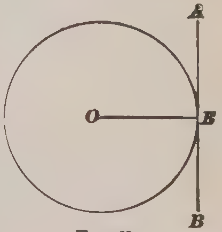


FIG. 68

75. A **tangent** to a circle is a straight line that touches the circle at one point only; it is always perpendicular to a radius drawn to that point. Thus, $A\,B$, Fig. 68, is a tangent to the circle; it touches the circle at E and is perpendicular to the radius $O\,E$.

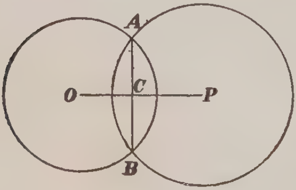


FIG. 69

76. If two circles intersect each other, the line joining their centers bisects at right angles the line joining the two points of intersection. Thus, if the two circles whose centers are O and P, Fig. 69, intersect at A and B, the line $O\,P$ bisects at right angles the line $A\,B$; or $A\,C = B\,C$.

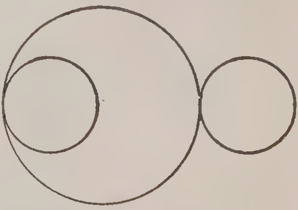


FIG. 70

77. One circle is said to be tangent to another circle when they touch each other at one point only, as in Fig. 70. This point is called the *point of contact*, or the *point of tangency*.

78. When two or more circles are described from the same center, as in Fig. 71, they are called *concentric circles.*

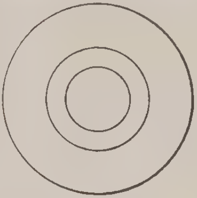


FIG. 71

79. If two chords of a circle intersect, the product of the segments (parts) thus formed of one of the chords is equal to the product of the segments of the other. Thus, $A\,B$ and $C\,D$, Fig. 72, are chords that intersect at E. Then,

$$A\,E \times E\,B = C\,E \times E\,D$$

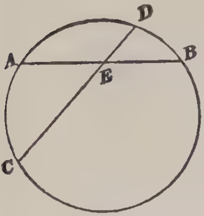


FIG. 72

EXAMPLE.—In Fig. 73, the diameter $B\,E$ of the circle is 29 feet, and the distance $B\,D$, 4 feet. What is the length of the chord $A\,C$ that passes through D and is perpendicular to $B\,E$?

SOLUTION.—Since $A\,C$ is perpendicular to the diameter $B\,E$, $B\,E$ bisects $A\,C$, and $A\,D = D\,C$ (Art. 73).

$$D\,E = 29 - 4 = 25 \text{ ft.}$$

The products of the segments of the intersecting chords being equal, $A\,D \times D\,C = B\,D \times D\,E$. But, $A\,D = D\,C$; therefore,

$$\overline{A\,D}^2 = B\,D \times D\,E = 4 \times 25 = 100;$$

$$A\,D = \sqrt{100} = 10 \text{ ft.}$$

and $$A\,C = 2 \times A\,D = 2 \times 10 = 20 \text{ ft.} \quad \text{Ans.}$$

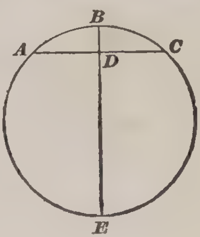


FIG. 73

80. Since $\overline{A\,D}^2 = B\,D \times D\,E$, in Fig. 73, the following principle may be deduced from the preceding solution: When a chord, as $A\,C$, is drawn perpendicular to a diameter of a circle, as $B\,E$, the square of one-half the chord is equal to the product of the parts of the diameter formed by the intersection of the chord. Here the chord $A\,C$ is divided into two equal parts, $A\,D$ and $D\,C$, by the diameter $B\,E$. Then, $\overline{A\,D}^2$, as well as $\overline{D\,C}^2$, is equal to $B\,D \times D\,E$.

EXAMPLE.—If, in Fig. 73, $B\,D$ is 3 feet and $D\,E$ is 9 feet, find the length of the half-chord $A\,D$.

SOLUTION.— $\overline{A\,D}^2 = B\,D \times D\,E = 3 \times 9 = 27$. Since the square of $A\,D = 27$,

$$A\,D = \sqrt{27} = 5.20 \text{ ft., or 5 ft. } 2\tfrac{3}{8} \text{ in., approximately.} \quad \text{Ans.}$$

81. In the circle $A\,B\,C\,E$, Fig. 74, $A\,C$ is the chord of the segment $A\,B\,C$. $B\,D$ is the height of the segment; $B\,C$, the chord of one-half the arc; and $B\,E$, the diameter perpendicular to the chord. Then,

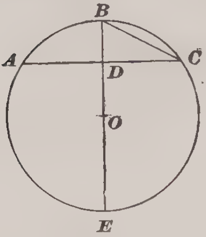


FIG. 74

$$\overline{B\,C}^2=\overline{B\,D}^2+\overline{D\,C}^2$$

But, $$\overline{D\,C}^2=B\,D\times D\,E$$

Therefore, $$\overline{B\,C}^2=\overline{B\,D}^2+B\,D\times D\,E$$

or, $$\overline{B\,C}^2=B\,D\,(B\,D+D\,E)$$

As $$B\,D+D\,E=B\,E,$$

$\overline{B\,C}^2=B\,D\times B\,E$, and $B\,E=\dfrac{\overline{B\,C}^2}{B\,D}$, or the radius $B\,O=\dfrac{\overline{B\,C}^2}{2\,B\,D}$

Rule.—*The radius of a circle is equal to the square of the chord of one-half of a given arc divided by twice the height of the segment.*

EXAMPLE.—Referring to Fig. 74, $B\,C=3$ feet 6 inches, and $B\,D=1$ foot 7 inches. Find the radius $B\,O$.

SOLUTION.— $B\,C=42$ in.; $B\,D=19$ in. According to the rule, the radius is

$$B\,O=\frac{\overline{B\,C}^2}{2\,B\,D}=\frac{42^2}{2\times 19}=46.42 \text{ in., or 3 ft. } 10\tfrac{7}{16} \text{ in., nearly. Ans.}$$

82. Where the chord of the arc and the height of the segment are known, to find the radius of the circle, add the square of one-half the chord to the square of the height of the segment and divide the sum by twice the height of the segment.

Let $r=$radius;
$c=$chord of arc;
$h=$height of segment.

Then, $$r=\frac{\left(\frac{c}{2}\right)^2+h^2}{2h}$$

or, $$r=\frac{c^2+4h^2}{8h}$$

EXAMPLE.—In arches, the *span* is the distance across the opening, as $A\,C$, Fig. 75, measured from the ends of the arch, as A and C. The *rise* $D\,B$ is the perpendicular distance from $A\,C$ to the highest point B measured on the center line $O\,B$.

The span $A\,C$, Fig. 75, of an arch is 6 feet, and the rise $D\,B$ is 8 inches. What is the length of the radius $O\,B$?

SOLUTION.—The span $A\,C$ is a chord of the arc $A\,B\,C$, and $B\,D$ is the height of the segment. $A\,D$, one-half of the chord, is $\frac{1}{2}\times 72$ in. $= 36$ in.

$$O\,B = \frac{\overline{A\,D}^2 + \overline{B\,D}^2}{2\,B\,D} = \frac{36^2+8^2}{2\times 8} = \frac{1360}{16} = 85 \text{ in.}$$

Therefore, $O\,B = 85$ in. $= 7$ ft. 1 in.
Ans.

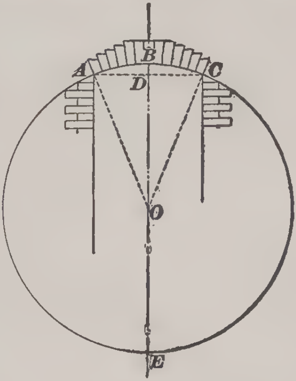


FIG. 75

INSCRIBED AND CIRCUMSCRIBED POLYGONS

83. An **inscribed polygon** is one whose vertexes lie on the circumference of a circle and whose sides are chords, as $K\,L\,M\,N\,P\,Q$, Fig. 76.

84. A **circumscribed polygon** is one whose sides are tangent to a circle, as $A\,B\,C\,D\,E\,F$, Fig. 76.

Circles may be inscribed in and circumscribed about any regular polygon. The radius of the circumscribed circle is the distance from the center to an angle of the inscribed polygon, and of the inscribed circle the perpendicular distance from the center to the side of the circumscribed polygon, as shown in Fig. 76.

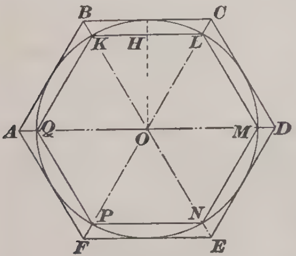


FIG. 76

85. If lines are drawn from the center to the angles of a regular polygon, they will make equal angles with the sides of the polygon and will form isosceles triangles. As the sum of all the angles formed about a point by lines drawn from it is equal to 360°, the angle at the center of a regular polygon between two radii equals 360° divided by the number of sides. Thus, the angle between two radii drawn to the ends of a side of a regular hexagon is $360° \div 6 = 60°$. The sum of the angles of a triangle is two right angles, or 180°; hence, since the triangles formed by drawing radii to the angles of a regular polygon are isosceles, the angles $O\,Q\,K$ and $O\,K\,Q$, Fig. 76, are each equal to one-half the difference between 180° and the central angle. Thus, each angle is $\frac{1}{2}\times(180°-60°)=60°$, and consequently the triangle $O\,K\,Q$ is equilateral. In a regular hexagon, therefore, the sides are equal to the radius of the circumscribed circle.

A line drawn from the center of a regular polygon to an angle bisects the angle; thus $O\,A$ bisects the angle $B\,A\,F$, which, therefore, is equal to $2\times O\,A\,B$, or $2\times O\,A\,F$.

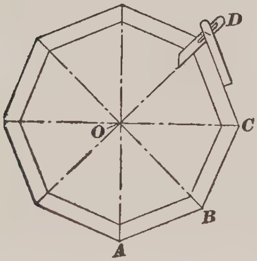


FIG. 77

86. The principles of Art. **85** are employed, in practice, to determine the various cuts of material used in the construction of regular polygons.

EXAMPLE 1.—Eight pieces of molding, as $A\,B$, $B\,C$, etc., Fig. 77, are to be laid around an octagonal room of equal sides, the ends of the pieces being sawed off at an angle so as to match, as shown. To what angle must a bevel, as shown at D, be set in order to mark the line of the saw cut?

SOLUTION.—Draw $A\,O$, $B\,O$, etc. to the center O, forming triangles $A\,O\,B$, $B\,O\,C$, etc. Then each of the angles at O is one-eighth of $360° = 45°$. The angles $O\,A\,B$ and $O\,B\,A$ are each

$$\tfrac{1}{2}\times(180°-45°)=67\tfrac{1}{2}°$$

If the blade of the bevel is set at this angle, as at D, and the ends of the pieces of molding are cut to it, they will fit together properly. Ans.

EXAMPLE 2.—In Fig. 78, $G\,H\,E\,F$ and $H\,K\,D\,E$ represent two boards forming a center for supporting a brick arch during building, the shaded parts being cut away, leaving a curve, as $A\,L\,B\,C$, on which to form the arch. What is (*a*) the length $G\,H$ (or $H\,K$) of each board, and (*b*) the angle of bevel $G\,H\,O$ where they meet? The radius $A\,O$ of the arch is 2 feet.

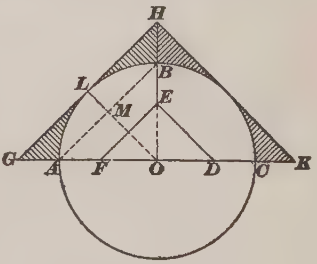


FIG. 78

Solution.—(*a*) Draw *A B* parallel to *G H* and meeting *O G* and *O H*; also draw *O L* perpendicular to *A B*; then *A M*=*M B* (Art. **73**), and by similar triangles, *L G*=*L H*. *O L* bisects the right angle *A O B*; hence, *A O M* is one-half of 90°, or 45°. Then the other acute angle in the right triangle *O M A* is 90°−45°=45° also. The angle *L G O* is equal to the angle *M A O*, or 45°, and the right triangle *O L G* having two angles equal, the opposite sides are equal also; and since *O L*=the radius, or 2 ft., *L G* is also 2 ft. and *G H* is 2×2=4 ft., the required length. Ans.

(*b*) Angle *G H O*=angle *O G H*=45°, the angle at which to set the bevel. Ans.

87. The following principle may also be conveniently used in many cases. In any polygon, the sum of all the angles equals 180° multiplied by a number that is two less than the number of sides in the polygon. Hence, to find the size of any one of the angles of a regular polygon, multiply 180° by the number of sides less two, and divide the result by the number of sides; the quotient will be the number of degrees in each angle. For example, each angle of a regular octagon, as in Fig. 77, is [180×(8−2)]÷8=135°.

EXAMPLES FOR PRACTICE

1. The span of a circular arch is 7 feet, and the rise is 1 inch for each foot of span. What is the radius of the arch?

Ans. 10 ft. 9½ in.

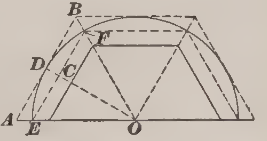


Fig. 79

2. The centering for an arch, Fig. 79, 8 feet in diameter, is made of three pieces of equal lengths. Knowing that the side of a regular inscribed hexagon is equal to the radius, what should be the length, as *A B*, of each piece?

Ans. 4.62 ft., or 4 ft. 7½ in., nearly

Suggestion.—Find the length of *O C* in the right triangle *O C E*; then by the similar triangles *O E C* and *O A D* find the length of *A O*. The proportion will be stated: *O C* : *O D*=*O E* : *O A*. *A B* will equal *A O*.

3. In the preceding example, what is: (*a*) the bevel angle *A B O*? (*b*) the angle between any two adjacent pieces?

Ans. (*a*) 60°; (*b*) 120°

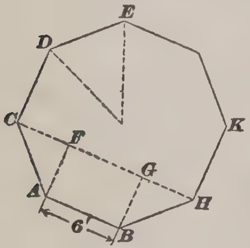


Fig. 80

4. The flooring in a regular octagonal room, Fig. 80, is laid parallel to the side *A B*, which is 6 feet long. What is the angle of bevel for cutting the ends of the pieces that fit along *A C*?

Ans. 135° or 45°

5. What is the length of the pieces of flooring between the sides *C D* and *H K*, Fig. 80, knowing that *C F* and *A F* are equal and that the angle *C F A* is a right angle? (See Art. **60.**)

Ans. 14.48 ft., or 14 ft. $5\frac{3}{4}$ in., nearly

6. In Fig. 73, if the diameter of the circle is 10 feet and *D E* is 8 feet 6 inches, find the distance *D C*.

Ans. 3 ft. $6\frac{7}{8}$ in., nearly

GEOMETRY AND MENSURATION

(PART 2)

Serial 1098B-3 Edition 2

FINDING AREAS AND VOLUMES

MENSURATION OF PLANE SURFACES

UNITS OF MEASUREMENT

1. The **area** of a figure means the extent of the surface included within the bounding lines of the figure, and is expressed by the number of unit squares it will contain.

2. A **unit square** is the square having the unit for its side. For example, if the unit is 1 inch, the unit square is the square whose side measures 1 inch in length. In measuring a surface, the area may be expressed by the number of square inches that the surface contains. If the unit were 1 foot, the unit square would measure 1 foot on each side, and a surface would then be measured by the number of square feet that it contained.

The square that measures 1 inch on a side is called a **square inch,** and the one that measures 1 foot on a side is called a **square foot.** Square inch and square foot are abbreviated to sq. in. and sq. ft., or are sometimes indicated by $\square''$ and $\square'$.

QUADRILATERAL

3. A **parallelogram** is a quadrilateral, or four-sided figure, whose opposite sides are parallel. There are four kinds of parallelograms: the *square*, the *rectangle*, the *rhombus*, and the *rhomboid*.

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4. A **rectangle,** Fig. 1, is a parallelogram whose angles are all right angles.

5. A **square,** Fig. 2, is a rectangle, all of whose sides are equal.

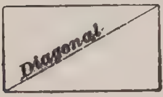


FIG. 1

FIG. 2

6. A **rhomboid,** Fig. 3, is a parallelogram whose opposite sides only are equal, and whose angles are not right angles.

7. A **rhombus,** Fig. 4, is a parallelogram having equal sides, and whose angles are not right angles.

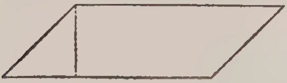


FIG. 3

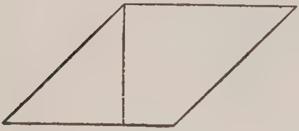


FIG. 4

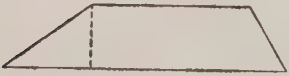


FIG. 5

8. A **trapezoid,** Fig. 5, is a quadrilateral that has only two of its sides parallel.

9. A **trapezium,** Fig. 6, is a quadrilateral having no two sides parallel.

10. The **altitude** of a parallelogram, or of a trapezoid, is the perpendicular distance between the parallel sides. In Figs. 3 to 5, the altitude is indicated by a dotted vertical line.

11. A **diagonal** is a straight line drawn from the vertex of any angle of a quadrilateral to the vertex of the angle opposite; a diagonal divides the quadrilateral into two triangles. A diagonal divides a parallelogram into two *equal* triangles.

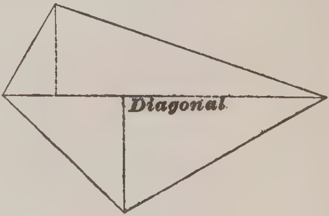


FIG. 6

12. The area of any parallelogram may be found by the following rule:

Rule.—*To find the area of any parallelogram, multiply the base by the altitude.*

Let A = area;

h = altitude;

b = base.

Then, $A = b\,h$

EXAMPLE 1.—What is the area of the rectangle $A\,B\,C\,D$, Fig 7?

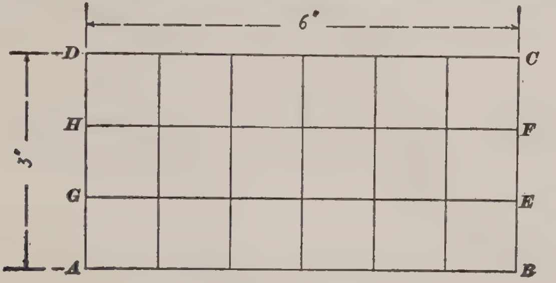


FIG. 7

SOLUTION.—The area of the rectangle equals the product of the base, which is 6 in., and the height, which is 3 in. Hence, the area is

$$6 \times 3 = 18 \text{ sq. in.} \quad \text{Ans.}$$

NOTE.—By dividing Fig. 7 into squares as shown, it is seen that, if the figure is drawn full size, it will require 18 square inches to cover it.

EXAMPLE 2.—A rectangular lot is 4 rods wide and 8 rods long: (*a*) How many square feet does it contain? (*b*) What part of an acre is it?

SOLUTION.—(*a*) Reducing rods to feet, 4 rd. = 66 ft., and 8 rd. = 132 ft. Either side may be taken as the base. Apply the formula, and

$$A = b\,h = 66 \times 132 = 8{,}712 \text{ sq. ft.} \quad \text{Ans.}$$

(*b*) As there are 43,560 sq. ft. in an acre, 8,712 sq. ft. = 8,712 ÷ 43,560 = .2 = $\frac{1}{5}$ A. Ans.

13. The area of a trapezoid may be found as follows:

Rule.—*To find the area of a trapezoid, multiply one-half the sum of the parallel sides by the altitude.*

Let a and b = lengths of parallel sides;

h = altitude.

Then,

$$A = h\left(\frac{a+b}{2}\right)$$

EXAMPLE.—How many square feet of floor space are there in a five-story warehouse, if each floor has the dimensions given in Fig. 8?

SOLUTION.—As either of the parallel sides may be called a, let it be the shorter; $a = 49$ ft. 6 in. $= 49.5$ ft.; $b = 54$ ft. 3 in. $= 54.25$ ft.; and $h = 27$ ft. 9 in. $= 27.75$ ft. Then, apply the formula, and

$$A = h\left(\frac{a+b}{2}\right) = 27.75\left(\frac{49.5+54.25}{2}\right) = 1{,}439.5 \text{ sq. ft.}$$

Therefore, the area of five such floors is $5 \times 1{,}439.5 = 7{,}197.5$ sq. ft. Ans.

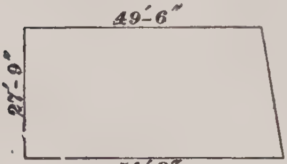
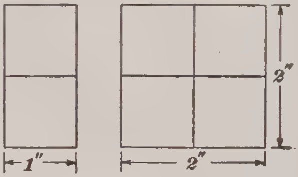


FIG. 8



(a) (b)

FIG. 9

14. It should be observed that *2 square inches* does not mean the same as *2 inches square; 3 square inches* does not mean the same as *3 inches square;* etc. Fig. 9 (a) shows a rectangle 2 inches by 1 inch, and its area is 2 square inches; (b) shows a square 2 inches by 2 inches; that is, a figure 2 inches square, and its area is 4 square inches.

TRIANGLE

15. The following rule may be employed in finding the area of a triangle:

Rule.—*To find the area of a triangle, multiply the base by the altitude and divide the product by 2.*

Let A = area;
b = base;
h = altitude.

Then, $$A = \frac{b\,h}{2}$$

The base and the altitude must always be stated in the same units; that is, both must be in inches, or both in feet, in which case the area found will be given in square inches, or square feet, respectively.

EXAMPLE.—The triangular gable of a house is 24 feet wide and 12 feet high. What is its area?

Solution.—The gable is a triangle having a base of 24 ft. and an altitude of 12 ft. Hence, according to the rule, the area is

$$A=(24\times 12)\div 2=144 \text{ sq. ft.} \quad \text{Ans.}$$

16. If the triangle is a right triangle, one of the short sides may be taken as the base, and the other short side as the altitude; hence, *the area of a right triangle is equal to one-half the product of the two short sides.*

17. If the lengths of the three sides of a triangle are given, the area may be found. Call the three sides a, b, and c, and let s represent half the sum of the sides, or $s=\frac{1}{2}(a+b+c)$. Then the area A of the triangle may be found by the formula

$$A=\sqrt{s(s-a)(s-b)(s-c)}$$

In other words, each side is subtracted from half the sum of the three sides, and these three remainders are multiplied together and by half the sum of the sides, or s; then the square root of the product is the area.

Example.—What is the area of a triangular plot of ground, the sides of the plot being 60, 55, and 35 feet, respectively?

Solution.—It does not matter which side is called a, b, or c. Letting $a=60$, $b=55$, and $c=35$,

$$s=\tfrac{1}{2}(60+55+35)=75$$

Then,

$$s-a=75-60=15$$
$$s-b=75-55=20$$
$$s-c=75-35=40$$

and

$$A=\sqrt{75\times 15\times 20\times 40}=\sqrt{900{,}000}=948.68 \text{ sq. ft.} \quad \text{Ans.}$$

POLYGONS

18. To find the area of any figure bounded by straight lines the following rule may be employed:

Rule.—*To find the area of any figure bounded by straight lines, divide the figure into triangles, parallelograms, and trapezoids, and find the area of each. The sum of these partial areas will be the area of the figure.*

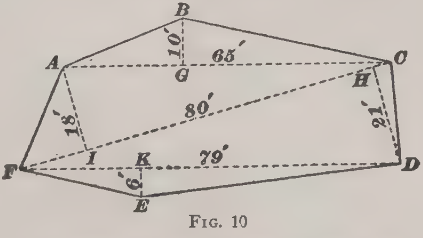


Fig. 10

EXAMPLE.—A lot has the dimensions given in Fig. 10. How many square feet does it contain?

SOLUTION.—The distances marked on Fig. 10 being measured. the partial areas are found to be:

$$\text{Triangle } A\,B\,C=\frac{65\times10}{2}=325 \text{ sq. ft.}$$

$$\text{Triangle } F\,A\,C=\frac{80\times18}{2}=720 \text{ sq. ft.}$$

$$\text{Triangle } F\,C\,D=\frac{80\times21}{2}=840 \text{ sq. ft.}$$

$$\text{Triangle } F\,D\,E=\frac{79\times6}{2}=\underline{237 \text{ sq. ft.}}$$

$$2{,}122 \text{ sq. ft.}$$

The area of the lot is therefore the sum of these, or 2,122 sq. ft. Ans.

19. A regular polygon may be divided into as many equal triangles as the figure has sides by drawing lines from the center of the polygon to its angles. In Fig. 11, for example, the hexagon $A\,B\,C\,D\,E\,F$ is composed of six equal triangles, $A\,O\,B$, $B\,O\,C$, $C\,O\,D$, $D\,O\,E$, $E\,O\,F$, and $F\,O\,A$. The area of each triangle is equal to one-half of the product of a side and the altitude, which is the perpendicular distance $O\,G$ from the center to a side. The entire area of the polygon is therefore obtained by multiplying the area of one of these triangles by the number of triangles, or six; or, the area of a regular polygon is found by multiplying one-half the perimeter of the figure by the perpendicular distance from the center to a side.

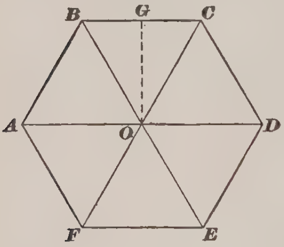


FIG. 11

EXAMPLE.—What is the area of an octagonal room, each of whose eight sides is 9 feet, when the distance from the center of the room to a side is 10 feet $10\frac{1}{2}$ inches?

SOLUTION.—The perimeter is $8\times9=72$ ft.; 10 ft. $10\frac{1}{2}$ in. $=10.875$ ft.

$$\text{Area}=\frac{10.875\times72}{2}=391.5 \text{ sq. ft.} \quad \text{Ans.}$$

20. The areas of regular polygons whose names are given in column B, of Table I, and whose sides are each 1,

TABLE I

TABLE OF REGULAR POLYGONS

Number of Sides	Name	Area, When Side = 1	Radius of Circumscribed Circle, When Side = 1	Side, When Radius of Circumscribed Circle = 1	Radius of Inscribed Circle, When Side = 1	Side, When Radius of Inscribed Circle = 1	Radius of Circumscribed Circle, When Radius of Inscribed Circle = 1	Angle at Each Vertex	One-Half of Angle at Vertex
A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*
3	Triangle	.4330	.5773	1.7321	.2887	3.4641	2	60°	30°
4	Square	1	.7071	1.4142	.5	2	1.4142	90°	45°
5	Pentagon	1.7205	.8507	1.1756	.6882	1.4531	1.2361	108°	54°
6	Hexagon	2.5981	1	1	.8660	1.1547	1.1547	120°	60°
7	Heptagon	3.6339	1.1524	.8678	1.0383	.9632	1.1099	128° 34′	64° 17′
8	Octagon	4.8284	1.3066	.7654	1.2071	.8284	1.0824	135°	67° 30′
9	Nonagon	6.1818	1.4619	.6840	1.3737	.7279	1.0642	140°	70°
10	Decagon	7.6942	1.6180	.6180	1.5388	.6498	1.0515	144°	72°
11	Undecagon	9.3656	1.7747	.5635	1.7028	.5873	1.0422	147° 16′	73° 38′
12	Dodecagon	11.1962	1.9319	.5176	1.8660	.5359	1.0353	150°	75°

or unity, are found in column C opposite the corresponding name. For example, the area of a regular decagon each of whose sides is 1 foot is 7.6942 square feet. The areas of regular polygons mentioned in the table may be obtained as follows:

Rule I.—*To find the area of a regular polygon, multiply the square of the side by the number in column C standing opposite the name of the polygon in column B.*

Example 1.—What is the area of a regular twelve-sided figure, each of its sides being 9 feet long?

Solution.—The figure is a dodecagon, and the corresponding number in column C is 11.1962.

$$\text{Area} = 9 \times 9 \times 11.1962 = 906.89 \text{ sq. ft.} \quad \text{Ans.}$$

Rule II.—*To find the side of a regular polygon when the area is known, divide the area by the number in column C corresponding to the name of the polygon and extract the square root of the result.*

Example 2.—What is the side of a regular octagon that contains an area of 2,000 square feet?

Solution.—The number in column C to be used for an octagon is 4.8284. $2,000 \div 4.8284 = 414.22$. The square root of 414.22 is 20.35. Hence, 20.35 ft., or 20 ft. $4\frac{1}{4}$ in., nearly, is the length of the side. Ans.

21. In column D of Table I, are given the radii of the circles that may be described about the regular polygons named, when each side is of unit length. In column E are given the lengths of sides of inscribed polygons when the radius of the circle is 1, etc. The use of the various columns will be made clear by the following rules and illustrative examples:

Rule I.—*To find the radius of the circumscribed circle when the side of a regular polygon is known, multiply the side by the number in column D opposite the name of the polygon.*

Example 1.—It is required to lay out a regular seven-sided figure, each side being 2 feet. What is the radius of the circle whose circumference will pass through each corner of the figure?

Solution.—The number in column D opposite heptagon is 1.1524. Radius of circumscribed circle is

$$2 \times 1.1524 = 2.3048, \text{ or, 2 ft. } 3\tfrac{5}{8} \text{ in., nearly.} \quad \text{Ans.}$$

Rule II.—*To find the length of the side of a regular polygon that may be inscribed in a circle of known radius, multiply the known radius by the number in column E opposite the name of the polygon.*

EXAMPLE 2.—What is the length of the side of an octagon that may be inscribed in a circle 12 feet in diameter?

SOLUTION.—Radius is $12 \div 2 = 6$ ft. Number in column E opposite octagon is .7654. Length of side is

$6 \times .7654 = 4.5924$, or 4 ft. $7\frac{1}{8}$ in., nearly. Ans.

Rule III.—*To find the radius of the inscribed circle when the side of a regular polygon is known, multiply the side by the number in column F opposite the name of the polygon.*

EXAMPLE 3.—It is desired to determine the perpendicular distance from the center of a regular nonagon to each of its sides, when the length of each side is 5 feet.

SOLUTION.—The perpendicular distance from the center of the polygon to its sides is equal to the radius of the circle that may be inscribed in the polygon. The number in column F opposite nonagon is 1.3737; hence, perpendicular distance is

$5 \times 1.3737 = 6.8685$, or 6 ft. $10\frac{7}{16}$ in., nearly. Ans.

Rule IV.—*To find the side of a regular polygon when the radius of the inscribed circle is known, multiply the radius by the corresponding number in column G.*

Rule V.—*To find the radius of the circle circumscribed about a regular polygon when the radius of the inscribed circle is known, multiply the given radius by the corresponding number in column H.*

22. In column *I* of Table I, are found the angles of each vertex of a regular polygon formed by the adjacent sides.

In column *J* are found the values of one-half the angle at each vertex, or the angles that should be used in making the bevel cuts for moldings or material laid parallel to the sides, and jointed.

EXAMPLES FOR PRACTICE

1. A five-pointed star is inscribed in the circle *A B C D E*, Fig. 12. The diameter of the circle is 4 feet 6 inches. What is the distance between consecutive points of the star measured on a straight line? Ans. 2 ft. $7\frac{3}{4}$ in., nearly

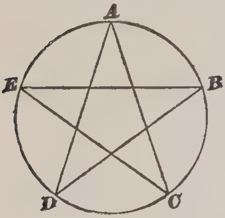


FIG. 12

2. Each side of an octagonal room is 12 feet 3 inches. Find the number of square feet of flooring required for the room, making no allowance for waste. Ans. 724.56 sq. ft.

3. What is the area of the end of a hexagonal rod, if the distance between parallel sides of the rod is 2 inches? Ans. 3.46 sq. in.

4. In Fig. 13, *A B C D* represents the end section of a piece of timber 6 inches square. (*a*) What will be the side of a regular octagon that may be formed from the stick, if the corners are cut off as shown? (*b*) What is the distance *B E*?

Ans. $\begin{cases} (a)\ 2\frac{1}{2}\text{ in.} \\ (b)\ 1\frac{3}{4}\text{ in.} \end{cases}$

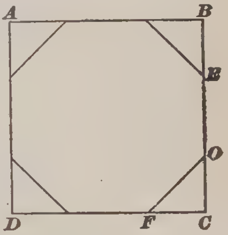


FIG. 13

CIRCLE

23. The ratio of the circumference of a circle to its diameter is usually denoted by the Greek letter π (pronounced *pie*). The approximate value of the ratio, correct to four decimal places, is 3.1416; hence, approximately, $\pi = 3.1416$. For off-hand calculations, the ratio is frequently taken as $3\frac{1}{7}$; that is, the circumference is $3\frac{1}{7}$ times the diameter.

24. The following rules are used to find the circumference, diameter, and radius of a circle:

Rule I.—*To find the circumference of a circle, multiply the diameter by 3.1416; or, multiply the radius by 2×3.1416.*

Rule II.—*To find the diameter of a circle, divide the circumference by 3.1416; or, divide the circumference by 2×3.1416 to find the radius.*

Let c = circumference of circle;
d = diameter of circle;
r = radius of circle;
$\pi = 3.1416$

Then, the preceding rules may be expressed by the formulas

$$c=\pi d=2\pi r \qquad (1)$$

$$d=\frac{c}{\pi} \qquad (2)$$

$$r=\frac{c}{2\pi} \qquad (3)$$

EXAMPLE.—A wheelwright wishes to cut a length of tire iron long enough to go around a $4\frac{1}{2}$-foot wagon wheel. How long a piece should be cut, if it is necessary to allow $1\frac{1}{2}$ inches for welding?

SOLUTION.—Here $d=4\frac{1}{2}$ ft.; so, apply formula **1**, thus:

$$c=\pi\,d=3.1416\times4\tfrac{1}{2}=14.14 \text{ ft.}=14 \text{ ft. } 1\tfrac{3}{4} \text{ in., nearly}$$

Adding the $1\frac{1}{2}$ in., the length required is 14 ft. $3\frac{1}{4}$ in. Ans.

25. The length of an arc of a circle may be found by the following rule:

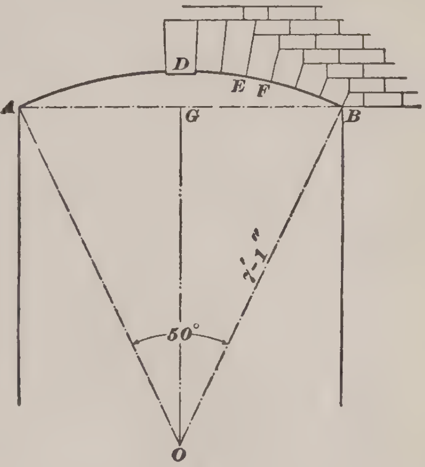


FIG. 14

Rule.—*To find the length of an arc of a circle, multiply the number of degrees in the arc by the radius, and divide by 57.3, or by 57.296 if greater precision is required.*

Let l = length of arc;
r = radius of arc;
n = number of degrees in arc.

Then,

$$l=\frac{r\,n}{57.3}$$

If the angle contains degrees, minutes, and seconds, reduce them to degrees and decimals of a degree before substituting the value of the angle in the formula; for example,

$$37^\circ\,30'\,15''=37^\circ+\tfrac{30}{60}^\circ+\tfrac{15}{3600}^\circ$$
$$=37^\circ+.5^\circ+.004^\circ=37.504^\circ$$

EXAMPLE.—In Fig. 14 is shown a part of a stone arch, the outline of the under surface of which is an arc $A\,D\,B$ of a circle having a radius of 7 feet 1 inch, or 85 inches. The angle $A\,O\,B$ is 50°. If there are thirteen stones in the arch, what will be the width, as *E F*, of each stone, measured along the arc?

Solution.—Here $l = ADB$, the length of the arc; $r = 85$ in., and $n = 50°$. Apply the formula, and

$$l = \frac{rn}{57.3} = \frac{85 \times 50}{57.3} = 74.17 \text{ in.}$$

Divide by the number of stones, and the width of each is $74.17 \div 13 = 5.71$ in. $= 5\frac{11}{16}$ in., nearly. Ans.

26. When only the chord *A B*, Fig. 15, of an arc and the height, or rise, *C D* of the segment are known, the following approximate method of finding the length of arc gives good results. *A C*, the chord of half the arc, has the value

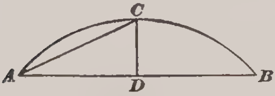


Fig. 15

$$AC = \sqrt{\overline{AD}^2 + \overline{CD}^2} = \sqrt{\left(\frac{AB}{2}\right)^2 + \overline{CD}^2}$$

Then, to find the length of the arc use the following rule:

Rule.—*From eight times the chord of one-half the arc, subtract the chord of the whole arc and divide the remainder by 3.*

That is, $$\text{arc } ACB = \frac{8 \times AC - AB}{3}$$

Let

c = chord of whole arc;
h = height of segment;
l = length of arc.

Then, $$AC = \sqrt{\frac{c^2}{4} + h^2} = \frac{1}{2}\sqrt{c^2 + 4h^2}$$

and $$l = \frac{4\sqrt{c^2 + 4h^2} - c}{3}$$

Either the rule or the formula may be used, whichever is the more convenient.

This method gives the length of an arc less than one-sixth of the circumference correct to four figures, and it gives the length of an arc less than one-third of the circumference correct to three figures.

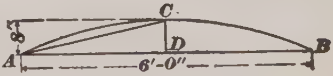


Fig. 16

Example.—Find the length of the arc $A\ C\ B$, Fig. 16.

Solution.—$A\ B = 6$ ft. $= 72$ in. $A\ D$, which is one half of $A\ B$ equals $\frac{1}{2} \times 72 = 36$ in.; and $D\ C = 8$ in.

$$A\ C = \sqrt{\overline{A\ D}^2 + \overline{D\ C}^2} = \sqrt{36^2 + 8^2} = 36.88 \text{ in.}$$

Hence, the length of the arc $A\ C\ B$ is

$$\frac{8 \times A\ C - A\ B}{3} = \frac{8 \times 36.88 - 72}{3} = 74.35 \text{ in., or 6 ft. } 2\tfrac{3}{8} \text{ in., nearly. Ans.}$$

27. When the results that may be obtained by the method given in the preceding article are not considered sufficiently accurate, the length of one-half the arc may be found in a similar manner and this multiplied by 2. The following example will fully illustrate the method of procedure:

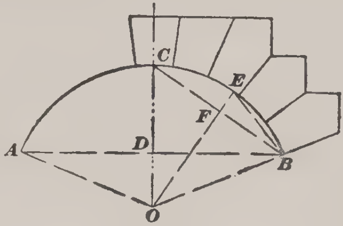


Fig. 17

Example.—The arch in Fig. 17 has a span $A\ B$ of 60 inches and a rise $C\ D$ of 25 inches. What is the length of the arc $A\ C\ B$?

Solution.—Let O represent the center of the arch, and let F be the middle point of the chord $C\ B$.

$$\overline{C\ B}^2 = \overline{C\ D}^2 + \overline{D\ B}^2 = 25^2 + \left(\tfrac{60}{2}\right)^2 = 1{,}525$$

$$C\ B = \sqrt{1{,}525} = 39.051 \text{ in.}$$

$$C\ F \text{ or } F\ B = C\ B \div 2 = 39.051 \div 2 = 19.526 \text{ in.}$$

The radius, according to a principle explained in a preceding Section is

$$C\ O = \frac{\overline{C\ B}^2}{2\ C\ D} = \frac{1{,}525}{2 \times 25} = 30.5 \text{ in.}$$

In the right triangle $C\ F\ O$,

$$O\ F = \sqrt{\overline{O\ C}^2 - \overline{C\ F}^2} = \sqrt{30.5^2 - 19.526^2} = 23.431 \text{ in.}$$

$$F\ E = O\ E - O\ F = 30.5 - 23.431 = 7.069 \text{ in.}$$

In the right triangle $E\ F\ B$,

$$E\ B = \sqrt{\overline{E\ F}^2 + \overline{F\ B}^2} = \sqrt{7.069^2 + 19.526^2} = 20.766 \text{ in.}$$

According to the rule of Art. **26**,

$$\text{arc } C\ E\ B = \frac{8\ E\ B - C\ B}{3} = \frac{8 \times 20.766 - 39.051}{3} = 42.359 \text{ in.}$$

The whole arc $A\ C\ B$ is therefore

$$2 \times 42.359 = 84.72 \text{ in., or 7 ft. } \tfrac{3}{4} \text{ in. Ans.}$$

If the rule of Art. **26**, is applied directly,

arc $A\,C\,B=\dfrac{8\times CB-AB}{3}=\dfrac{8\times 39.051-60}{3}=84.14$ in., or 7 ft. $\frac{1}{8}$ in. approximately. Ans.

A difference of results, as in this case, is to be expected when answers are obtained by different methods.

28. If the radius of an arc and the length of the chord are given, the height of the segment may be found. In Fig. 17, for example, suppose that the chord $A\,B$ and the radius $O\,A$ are known, and that the height $C\,D$ is to be found. As $A\,C\,B$ is an arc of a circle, $O\,C=O\,A$. But, $O\,C=C\,D+D\,O$; or, $C\,D=O\,C-D\,O$. Now, $D\,O$ is one side of a right triangle $A\,D\,O$, the hypotenuse being $O\,A$ and the other side being $A\,D=\frac{1}{2}A\,B$. Therefore,

$$\overline{D\,O}^2=\overline{O\,A}^2-\overline{A\,D}^2$$

or,

$$\overline{D\,O}^2=\overline{O\,A}^2-(\tfrac{1}{2}\overline{A\,B})^2$$

from which

$$D\,O=\sqrt{\overline{O\,A}^2-\tfrac{1}{4}\overline{A\,B}^2}$$

Then,

$$C\,D=O\,C-\sqrt{\overline{O\,A}^2-\tfrac{1}{4}\overline{A\,B}^2}$$

or,

$$C\,D=O\,A-\sqrt{\overline{O\,A}^2-\tfrac{1}{4}\overline{A\,B}^2} \qquad (1)$$

If the radius $O\,A$ and the height $C\,D$ are known, the length of the chord $A\,B$ may be found by the formula

$$A\,B=2\sqrt{\overline{O\,A}^2-(O\,A-C\,D)^2} \qquad (2)$$

29. The area of a sector of a circle may be found by the following rule:

Rule.—*To find the area of a sector of a circle, divide the number of degrees in the arc of the sector by 360. Multiply the result by the area of the circle of which the sector is a part.*

Let a = area of sector;
n = number of degrees in arc;
A = area of circle;
r = radius of circle.

Then,

$$a=\frac{n\,A}{360}=.0087267\,n\,r^2$$

Example.—A circular window 4 feet in diameter is divided by radial ribs into twelve equal sections. What is the area of each space?

Solution.—Here $n=\frac{360°}{12}=30°$; so apply the formula, and

$$a=\frac{n\,A}{360}=\frac{30\times .7854\times 4^2}{360}=1.05 \text{ sq. ft.} \quad \text{Ans.}$$

Or, $a=.0087267\,n\,r^2=.0087267\times 30\times 4=1.05$ sq. ft. Ans.

30. The area may also be found by the following rule:

Rule.—*To find the area of a sector, multiply one-half of the length of the arc by the radius.*

Let a = area of sector;
l = length of arc;
r = radius of arc.

Then, $a=\frac{1}{2}\,l\,r$

Example.—In the sector $O\,A\,C\,B$, Fig. 18, the radius $O\,A$ $(=O\,C)$ of the arc is 6 inches, and the length of the chord $A\,B$ is 7 inches. What is the area of the sector $O\,A\,C\,B$?

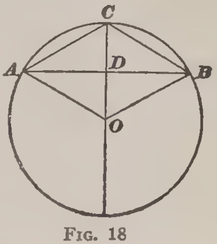


Fig. 18

Solution.—In order to find the area, it will be necessary first to find the length of the arc, by applying the rule given in Art. **26**; but before applying this rule, the height $C\,D$ of the segment will have to be found. Since $A\,D=\frac{1}{2}\,A\,B=\frac{1}{2}\times 7=3.5$,

$$O\,D=\sqrt{\overline{O\,A}^2-\overline{A\,D}^2}=\sqrt{6^2-3.5^2}=4.87 \text{ in.}$$

Then, $C\,D=O\,C-O\,D=6-4.87=1.13$ in.

$$C\,B=\sqrt{\overline{C\,D}^2+\overline{D\,B}^2}=\sqrt{1.13^2+3.5^2}=3.68 \text{ in.}$$

and $\text{arc } A\,C\,B=\frac{8\times C\,B-A\,B}{3}=\frac{8\times 3.68-7}{3}=7.48$ in.

Now apply the formula given in this article, and

$$a=\tfrac{1}{2}\,l\,r=\tfrac{1}{2}\times 7.48\times 6=22.44 \text{ sq. in.} \quad \text{Ans.}$$

31. The area of a segment of a circle may be found, approximately, as follows:

Rule.—*To two-thirds of the product of the height of the segment and the chord add the quotient obtained by dividing the cube of the height of the segment by twice the chord.*

Let A = area of segment;
c = chord;
h = height of segment.

Then,

$$A = \tfrac{2}{3}ch + \frac{h^3}{2c}$$

This rule will give results correct to three figures when the height of segment does not exceed the length of one-fourth of the chord, and the error will be less than 1 per cent. in finding the area of a semicircle. The rule is therefore sufficiently accurate for most ordinary calculations, as in estimating quantities and costs.

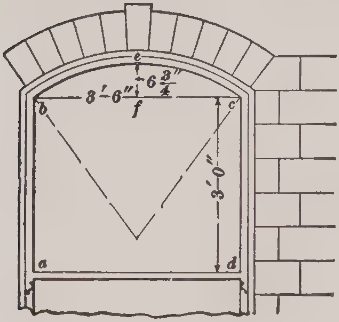


Fig. 19

Example.—What is the area of the upper pane of glass *a b e c d* in the window shown in Fig. 19?

Solution.—The glass may be divided into a rectangle *a b c d* 3 ft. × 3 ft. 6 in. and a segment *b e c* of a circle, the length of its chord *b c* being 3 ft. 6 in. and the height *e f* of segment 6.75 in. But, 3 ft. = 36 in., and 3 ft. 6 in. = 42 in. The area of the rectangle is, therefore, 42 × 36 = 1,512 sq. in. Referring to the segment, c = 3 ft. 6 in. = 42 in., and h = 6.75 in. Substitute in the formula, and

$$A = \tfrac{2}{3} \times 42 \times 6.75 + \frac{6.75^3}{2 \times 42}$$

$$= 189 + \frac{307.55}{84} = 189 + 3.66$$

$$= 192.66 \text{ sq. in.}$$

The entire area of the glass is

1,512 + 192.66 = 1,704.66 sq. in. Ans.

32. The exact area of a segment of a circle may be found by the following rule:

Rule.—*To find the area of a segment of a circle, find the area of the sector of which the segment is a part, and from this area subtract the area of the triangle formed by drawing radii to the extremities of the chord of the segment. The result is the area of the segment.*

Example.—What is the area of the segment $A\,C\,B$, Fig. 18?

Solution.—The area of the segment is equal to the difference between the areas of the sector $O\,A\,C\,B$ and the triangle $O\,A\,B$. The area of $O\,A\,C\,B$ has been found, in the solution to the example in Art. **30**, to be 22.44 sq. in. Area of triangle $O\,A\,B$ is

$$\tfrac{1}{2}\times O\,D\times A\,B=\tfrac{1}{2}\times 4.87\times 7=17.05 \text{ sq. in.}$$

Area of segment $A\,C\,B$ is, therefore,

$$22.44-17.05=5.39 \text{ sq. in.} \quad \text{Ans.}$$

33. The following rule is employed to find the area of a circle:

Rule.—*To find the area of a circle, square the diameter and multiply by .7854; or, square the radius and multiply by 3.1416.*

Let A = area of circle;
d = diameter of circle;
r = radius of circle.

Then,

$$A=\tfrac{1}{4}\pi d^2=.7854d^2 \qquad (1)$$

$$A=\pi r^2=3.1416r^2 \qquad (2)$$

Example 1.—What is the area of a circle whose radius is 14 inches?

Solution.—Apply formula **2**, and

$$A=3.1416\times 14^2=3.1416\times 196=615.75 \text{ sq. in.} \quad \text{Ans.}$$

Example 2.—The contract for a brick warehouse provides that all openings over 4 feet wide shall be deducted. What must be deducted for a semicircular arch 4 feet 8 inches in diameter?

Solution.—Here $d=4$ ft. 8 in. $=4\frac{2}{3}$ ft. Apply formula **1**, and

$$A=.7854d^2=.7854\times(4\tfrac{2}{3})^2=.7854\times 21.78=17.11 \text{ sq. ft.}$$

The area of the half circle is

$$17.11\div 2=8.56 \text{ sq. ft., or practically } 8\tfrac{1}{2} \text{ sq. ft.} \quad \text{Ans.}$$

34. When the area of a circle is given, the diameter may be found by applying the following rule:

Rule.—*To find the diameter of a circle, the area being given, divide the area by .7854 and extract the square root of the quotient.*

$$d=\sqrt{\frac{A}{.7854}}=\sqrt{\frac{4A}{\pi}}$$

Example.—To supply a certain quantity of water, it is necessary to have a pipe with an area of 28 square inches. What must be the diameter of the pipe?

SOLUTION.—Apply the formula just given, and

$$d=\sqrt{\frac{A}{.7854}}=\sqrt{\frac{28}{.7854}}=\sqrt{35.65}=5.97 \text{ in.}$$

As this is almost 6 in., a pipe of that diameter would be chosen. Ans.

35. The area of a flat circular ring, Fig. 20, may be found as follows:

Rule.—*To find the area of a flat circular ring, subtract the area of the smaller circle from that of the larger.*

Let A = area of ring;
d = outside diameter;
d_1 = inside diameter.

Then, $A=.7854d^2-.7854d_1^2=.7854\ (d^2-d_1^2)$

EXAMPLE.—What is the sectional area of a brick stack whose external and internal diameters are 6 feet 6 inches and 4 feet, respectively?

SOLUTION.—Here $d=6.5$ ft., and $d_1=4$ ft.; so, apply the formula and,

$$A=.7854\times(6.5^2-4^2)=.7854\times 26.25=20.62 \text{ sq. ft.} \quad \text{Ans.}$$

If one diameter and the area of the ring are known, the other diameter may be found as follows: Compute the area of the circle whose diameter is given and add to or subtract from this result the area of the ring. The sum or difference thus found will be the area of the circle whose diameter is required.

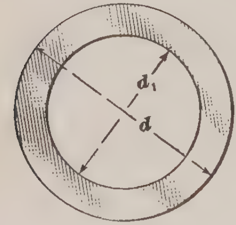


FIG. 20

If the outside diameter d and the width of the ring are given, and the area is to be found, multiply the width by 2 and subtract the product from the value of d. The remainder will be the value of d_1. Then, knowing d and d_1, the area may be found by the rule just given.

ELLIPSE

36. An **ellipse** is a plane figure bounded by a curved line, to any point of which the sum of the distances from two fixed points within, called the *foci*, is equal to the sum of the distances from the foci to any other point on the curve.

The curve bounding the ellipse is called the *circumference*, or *perimeter*, of the ellipse.

In Fig. 21, let A and B be the foci, and let C and D be any two points on the perimeter. Then, according to the definition just stated,

$$AC+CB=AD+DB$$

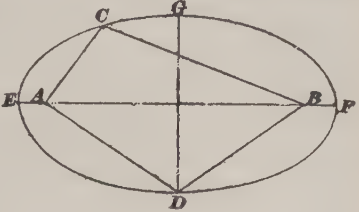


FIG. 21

Each of these sums is also equal to the major axis EF. The long diameter EF is called the *major axis;* the short diameter GD, the *minor axis.* The foci may be located from D or G as a center, and radius $DA=DB=\frac{1}{2}EF$.

37. There is no simple and exact method of finding the perimeter of an ellipse. The following formula gives values very nearly exact:

Let $C=$ perimeter;

$a=$ half the major axis;

$b=$ half the minor axis;

$$D=\frac{a-b}{a+b}.$$

Then,
$$C=\pi(a+b)\frac{64-3D^4}{64-16D^2}$$

EXAMPLE.—What is the perimeter of an ellipse whose axes are 12 inches and 8 inches long?

SOLUTION.—In this example, $a=6$ and $b=4$; hence, $D=\frac{6-4}{6+4}=\frac{2}{10}=.2$.

Then,
$$C=\pi(6+4)\frac{64-3(.2)^4}{64-16(.2)^2}$$

The only difficulty in simplifying this expression will be in reducing the fraction $\frac{64-3(.2)^4}{64-16(.2)^2}$ to its simplest form.

The numerator is simplified by raising .2 to the fourth power and subtracting three times the power from 64. Thus,

$$.2^4=.2\times.2\times.2\times.2=.0016;\ 3\times.0016=.0048$$

Then,
$$64-3(.2)^4=64-.0048=63.9952.$$

The denominator is simplified by multiplying the second power of .2 by 16 and subtracting the product from 64. Thus,

$$.2^2=.2\times.2=.04;\ 16\times.04=.64;\ 64-.64=63.36$$

The fraction therefore equals $\frac{63.9952}{63.36} = 1.010$. Consequently,

$$C = 3.1416 \times 10 \times 1.010 = 31.73 \text{ in., or 2 ft., } 7\tfrac{3}{4} \text{ in., nearly.} \quad \text{Ans.}$$

38. The following rule gives the method of finding the area of an ellipse:

Rule.—*To find the area of an ellipse, multiply the product of its two diameters by .7854.*

Let A = area of ellipse;
D = long diameter;
d = short diameter.

Then, $A = \frac{1}{4}\pi dD = .7854dD$

Example.—What is the area of the elliptic top of a table whose long and short diameters are 4 and 3 feet, respectively?

Solution.—Here $D = 4$ ft. and $d = 3$ ft.; so, apply the formula, and

$$A = .7854\, D\, d = .7854 \times 4 \times 3 = 9.42 \text{ sq. ft.} \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

1. Find the area of a square whose side is 5 feet 9 inches.
Ans. 33.06 sq. ft.

2. Find the area of a rhombus whose side is 12 feet 6 inches, and whose height is 9 feet 3 inches. Ans. 115.62 sq. ft.

3. How many square feet are there in a board 12 feet long, 18 inches wide at one end, and 12 inches wide at the other end? Ans. 15 sq.ft.

4. Find the distance around the outside of a waterwheel whose outside diameter is 22 feet 8 inches. Ans. 71 ft. $2\frac{1}{2}$ in.

5. The wheel of a carriage is observed to turn 375 times in going from a certain place to another. If the diameter of the wheel is 3 feet 6 inches, what is the distance between the two places?
Ans. 4,123 ft. $4\frac{1}{4}$ in., approximately

6. A circular column measures $45\frac{1}{2}$ inches around the outside. What is its diameter? Ans. $14\frac{1}{2}$ in., nearly

7. A belt covers an arc of 50° on a pulley whose diameter is 5 feet. What length of belt is in contact with the pulley?
Ans. 2 ft. $2\frac{3}{16}$ in., nearly

8. Find the area of a triangle whose three sides are 13, 14, and 15 feet.
Ans. 84 sq. ft.

9. Find the area of a right triangle whose hypotenuse is 50 feet and one of whose sides is 40 feet. Ans. 600 sq. ft

10. The parallel sides of a trapezoid are 321 feet 6 inches, and 214 feet 3 inches, and the perpendicular distance between them is 171 feet $1\frac{1}{2}$ inches. What is the area of the trapezoid? Ans. 45,840 sq. ft.

11. One side of a room is 16 feet long. If the floor contains 240 square feet, what is the width of the room? Ans. 15 ft.

12. The cable of a suspension bridge measures 40 inches in circumference. Find: (*a*) the diameter of the cable; (*b*) the area of the cross-section.

$$\text{Ans.}\begin{cases}(a)\ 12\frac{3}{4}\text{ in., nearly}\\(b)\ 127.3\text{ sq. in.}\end{cases}$$

13. In Fig. 15, if $AB=40$ inches and $CD=7\frac{1}{2}$ inches, find (*a*) the length of the arc ACB and (*b*) the radius of the arc.

$$\text{Ans.}\begin{cases}(a)\ 3\text{ ft. }7\frac{5}{8}\text{ in., approximately}\\(b)\ 2\text{ ft. }6\frac{7}{16}\text{ in., nearly}\end{cases}$$

14. (*a*) What is the perimeter of an ellipse whose axes are 15 inches and 9 inches? (*b*) What is the area?

$$\text{Ans.}\begin{cases}(a)\ 3\text{ ft. }2\frac{1}{4}\text{ in., approximately}\\(b)\ 106.03\text{ sq. in.}\end{cases}$$

MENSURATION OF SOLIDS

DEFINITIONS

39. A **solid,** or body, has three dimensions: length, breadth, and thickness. The sides that enclose it are called the **faces,** and their intersections are called **edges.**

40. The **entire surface** of a solid is the area of the whole outside of the solid, including the ends.

The **convex,** or **lateral, surface** of a solid is the same as the entire surface, except that the areas of the ends are not included.

41. The **volume** of a solid is the space included between its bounding surfaces, and is expressed by the number of times it will contain another volume, called the unit of volume Instead of the word *volume*, the expression **cubic contents** is frequently used.

PRISM AND CYLINDER

42. A **prism** is a solid whose ends are equal parallel polygons and whose sides are parallelograms. Prisms take their names from their bases. Thus, a triangular prism is one whose bases are triangles; a pentagonal prism is one whose bases are pentagons, etc.

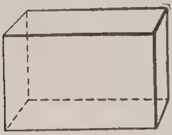


FIG. 22

43. A **parallelopiped,** as shown in Fig. 22, is a prism whose bases (ends) are parallelograms.

44. A **cube,** as shown in Fig. 23, is a parallelopiped whose faces and ends are squares.

45. The cube whose edges are equal to the unit of length is taken as the unit of volume when finding the volume of a solid. Thus, if the unit of length is 1 inch, the unit of volume will be the cube whose edges measure 1 inch, or 1 *cubic inch;* and the number of cubic inches the solid contains will be its volume. If the unit of length is 1 foot, the unit of volume will be 1 *cubic foot*, etc. Cubic inch, cubic foot, and cubic yard are abbreviated to cu. in., cu. ft., and cu. yd., respectively.

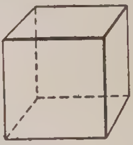


FIG. 23

46. A **cylinder,** shown in Fig. 24, is a solid whose ends are parallel and equal *curved* figures, and whose cross-section is uniform throughout its length. The *axis* of a cylinder is a straight line joining the centers of the ends, as shown. A *circular cylinder* is one any section of which, perpendicular to the axis, is a circle. Unless otherwise expressed the word *cylinder* always means a *circular cylinder.*

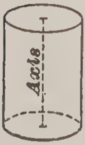


FIG. 24

47. A *right prism*, or *right cylinder*, is one whose axis is perpendicular to its base.

48. The *altitude* of a prism or cylinder is the perpendicular distance between its two ends.

49. The area of the convex surface of any right prism or right cylinder may be found according to the following rule:

Rule.—*To find the area of the convex surface of any right prism or right cylinder, multiply the perimeter of the base by the altitude.*

Let S = convex surface;
p = perimeter of base;
h = altitude.

Then, $S = p\,h$

To find the entire area, add the areas of the two ends to the convex area.

EXAMPLE 1.—A circular iron chimney, 7 feet in diameter and 85 feet high, is to be covered outside with a preservative paint. What will it cost to paint the chimney at 20 cents per square yard?

SOLUTION.—The perimeter p of the base is $\pi\,d = 21.99$ ft., and $h = 85$ ft. The surface of the chimney is $21.99 \times 85 = 1{,}869.2$ sq. ft., or 207.7 sq. yd. The cost of painting at 20 cents per sq. yd. is therefore

$$207.7 \times \$.20 = \$41.54. \quad \text{Ans.}$$

EXAMPLE 2.—How many square feet of zinc will be required to line a refrigerator having interior dimensions of 4 ft. × 6 ft. × 8 ft. high?

SOLUTION.—The area of the sides is

$$(4 \times 2 + 6 \times 2) \times 8 = 160 \text{ sq. ft.}$$

Adding for floor and ceiling

$(4 \times 6) \times 2 = 48$ sq. ft.; the total is 208 sq. ft. Ans.

EXAMPLE 3.—What will be the cost, at 35 cents per square yard, of plastering the walls and ceiling of an octagonal room, having sides 5 feet long and 11 feet high, the distance between the parallel sides being 12 feet?

SOLUTION.—The area of the sides is $8\ (5 \times 11) = 440$ sq. ft.; the area of the ceiling is one-half the perimeter times one-half the distance between the parallel sides, or $\frac{8 \times 5 \times 6}{2} = 120$ sq. ft.; the total is 560 sq. ft., or 62.22 sq. yd. At 35 ct. per sq. yd., the cost is \$21.78. Ans.

50. The volume of a rectangular solid may be found as follows:

Rule.—*To find the volume of a rectangular solid, multiply together the length, breadth, and height.*

Let V = volume;
l = length;
b = breadth;
h = height.

Then, $V = l\,b\,h$

Example.—Find the volume of the rectangular block shown in Fig. 25.

Solution.—The length is 4 in., the breadth is 3 in., and the height is 6 in. The volume therefore is

$$V=4\times3\times6=72 \text{ cu. in.} \quad \text{Ans.}$$

The block may be divided into six pieces, each 4 in. long, 3 in. wide, and 1 in. thick. Each of these pieces may be cut up into twelve 1-in. cubes. There will, therefore, be 6×12=72 one-inch cubes, or 72 cu. in. in the entire block.

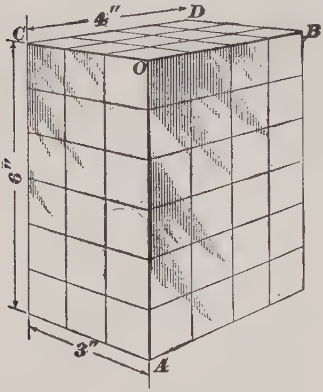


Fig. 25

51. The volume of a right prism, or cylinder, may be found by the following rule:

Rule.—*To find the volume of a right prism, or cylinder, multiply the area of the base by the altitude.*

Let V = volume;
a = area of base;
h = altitude.

Then, $V=a\,h$

If the given prism is a cube, the three dimensions are all equal, and the volume is equal to the cube of one of the edges.

If the volume and area of the base are given, the altitude is $\frac{V}{a}$. If a cylinder is hollow, the area of the base is equal to the area of a flat ring like that shown in Fig. 20.

Example 1.—If brickwork averages 21 bricks per cubic foot, how many bricks will there be in a pier 18 inches square and 6 feet high?

Solution.—The sectional area or a is $1.5\times1.5=2.25$ sq. ft.; $h=6$ ft.; hence,

$$V=a\,h=2.25\times6=13.5 \text{ cu. ft.}$$

At 21 bricks per cubic foot, the number of bricks in the pier is

$$13.5\times21=283.5. \quad \text{Ans.}$$

Example 2.—If cast iron weighs .26 pound per cubic inch, what length of sash weight will be necessary to balance a window weighing 8 pounds, the diameter of the cylindrical weight being $1\frac{1}{2}$ inches?

Solution.—As there are two weights, the weight of each is 4 lb. The number of cubic inches required, or V, is $4 \div .26 = 15.38$. The area a of a $1\frac{1}{2}$-in. circle is 1.77 sq. in.; hence, h, the length, is

$$\frac{V}{a} = \frac{15.38}{1.77} = 8.69 \text{ in., or } 8\tfrac{11}{16} \text{ in.} \quad \text{Ans.}$$

BOARD MEASURE

52. In measuring sawed lumber, the unit of measure is the **board foot,** and it is equal to the contents of a board 1 foot square and 1 inch thick. A board foot is, therefore, equal to one-twelfth of a cubic foot. Boards less than 1 inch thick are usually reckoned as though the thickness were 1 inch.

Rule I.—*To find the number of board feet in a piece of lumber, multiply together the length, in feet, the width, in feet, and the thickness, in inches.* Or,

Rule II.—*Multiply together the length, in feet, the width, in inches, and the thickness, in inches, and divide the product by 12.*

Example 1.—How many board feet are there in a plank 10 feet 6 inches long, 15 inches wide, and 3 inches thick?

Solution.—Length $= 10$ ft. 6 in. $= 10\frac{6}{12}$ ft. $= 10\frac{1}{2}$ ft. $= 10.5$ ft., width $= \frac{15}{12}$ ft. $= 1.25$ ft. The number of feet equals

$$10.5 \times 1.25 \times 3 = 39.375, \text{ or } 39.4$$

The result is, therefore, 39.4 ft. B. M. (board measure). Ans.

Example 2.—Find the number of board feet in twenty pieces of siding $\frac{1}{2}$ inch thick, $5\frac{1}{2}$ inches wide, and 10 feet 9 inches long.

Solution.—Since the thickness is less than 1 in., the boards are considered to be 1 in. thick when finding the amount of material in them. The calculation can be more easily performed in this case by using the second rule. Length $= 10\frac{9}{12}$ ft. $= 10.75$ ft., width $= 5\frac{1}{2}$ in. $= 5.5$ in.

$$\frac{20 \times 10.75 \times 5.5 \times 1}{12} = \frac{1{,}182.5}{12} = 98.54 \text{ ft. B. M.} \quad \text{Ans.}$$

PYRAMID AND CONE

53. A **pyramid** is a solid whose base is a polygon and whose sides are triangles uniting at a common point, called the **vertex.** A square pyramid is shown in Fig. 26.

54. A **cone** Fig. 27, is a solid whose base is a circle and whose convex surface tapers uniformly to a point called the *vertex.*

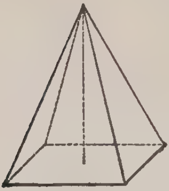


Fig. 26

55. A **right pyramid** or **cone** is one whose axis is perpendicular to the base.

56. The **altitude** of a pyramid or cone is the perpendicular distance from the vertex to the base.

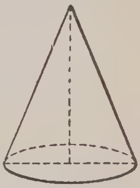


Fig. 27

57. The **slant height** of a *pyramid* is a line drawn from the vertex perpendicular to one of the sides of the base. It is the hypotenuse of a right triangle, one of whose sides is the height of the pyramid and the other is half the width of the base. The slant height of a *cone* is any straight line drawn from the vertex to the circumference of the base. It is the hypotenuse of a right triangle, one of whose sides is the height of the cone and the other is half the diameter of the base.

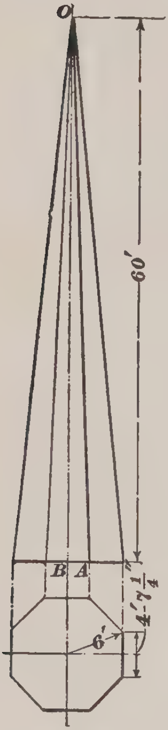


Fig. 28

58. The convex area of a right pyramid or cone may be found as follows:

Rule.—*To find the convex area of a right pyramid or cone multiply the perimeter of the base by one-half of the slant height.*

Let A = convex area;
p = perimeter of base;
s = slant height.

Then,
$$A=\frac{ps}{2}$$

Example.—A church steeple, in the form of an octagonal pyramid, as shown in Fig. 28, is 60 feet high. The outer radius of the base, or the radius of the circumscribed circle, is 6 feet, and the length of one of the sides of the base is 4 feet $7\frac{1}{4}$ inches. Find the convex surface.

SOLUTION.—To find the slant height, the length $O A$ along one edge must first be found. This is the hypotenuse of a right triangle of which the altitude is the height of the tower, or 60 ft., and the base is the radius, or 6 ft. Hence, $O A=\sqrt{60^2+6^2}=60.3$ ft. Now, in the triangle $O A B$, $O A$ is 60.3 ft. and $A B$ is half the length of one side of the octagonal base; but 4 ft. $7\frac{1}{4}$ in. $=4.6$ ft., approximately, and so $A B=4.6\div 2=2.3$ ft. The slant height $O B$ is, therefore, $\sqrt{60.3^2-2.3^2}=60.3$ ft., nearly. The perimeter of the base is $4.6\times 8=36.8$ ft. Apply the formula, and

$$A=\frac{p\,s}{2}=\frac{36.8\times 60.3}{2}=1{,}109.5 \text{ sq. ft.}\quad \text{Ans.}$$

59. The following rule states how to find the volume of a pyramid or cone:

Rule.—*To find the volume of a pyramid or cone, multiply the area of the base by one-third of the altitude.*

Let V = volume;
a = area of base;
h = altitude.

Then, $$V=\frac{ah}{3}$$

EXAMPLE 1.—The granite capstone of a monument is a square pyramid having a base 4 feet square and an altitude of 6 feet. If granite weighs 170 pounds per cubic foot, what will be the freight charges on the piece at 20 cents per 100 pounds?

SOLUTION.—The area a of the base is $4\times 4=16$ sq. ft., and the altitude h is 6 ft. Apply the formula, and

$$V=\frac{a\,h}{3}=\frac{16\times 6}{3}=32 \text{ cu. ft.}$$

At 170 lb. per cu. ft., the weight is $32\times 170=5{,}440$ lb. $=54.4$ cwt. The freight is, therefore,

$$54.4\times \$.20=\$10.88.\quad \text{Ans.}$$

EXAMPLE 2.—If, in the preceding example, the capstone were conical, having a base 4 feet in diameter and a height of 6 feet, how much less would it weigh than the pyramidal stone?

SOLUTION.—The area of the base is $.7854\ d^2=12.57$ sq. ft. Apply the formula, and the volume is

$$\frac{12.57\times 6}{3}=25.14 \text{ cu. ft.}$$

and the weight at 170 lb. per cu. ft. is 4,274 lb., nearly. As the pyramidal capstone weighs 5,440 lb., the difference in weight is

$$5{,}440-4{,}274=1{,}166 \text{ lb.}\quad \text{Ans.}$$

FRUSTUM OF PYRAMID OR CONE

60. If a pyramid or a cone is cut by a plane parallel to the base, so as to form two parts, the lower part is called the **frustum** of the pyramid or cone. The lower part in Fig. 29 is a frustum of a pyramid, and the lower part in Fig. 30 is a frustum of a cone.

The upper end of the frustum of a pyramid or cone is called the *upper base*, and the lower end the *lower base*. The altitude of a frustum is the perpendicular distance between the bases.

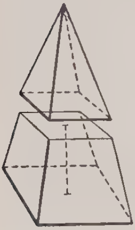


FIG. 29

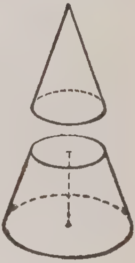


FIG. 30

61. The convex area of a frustum of a right pyramid or right cone is found as follows:

Rule.—*To find the convex area of a frustum of a right pyramid or right cone, multiply one-half the sum of the perimeters of the bases by the slant height.*

Let A = convex area;
P = perimeter of lower base;
p = perimeter of upper base;
s = slant height.

Then,

$$A = \frac{(P+p)s}{2}.$$

To find the entire area, add to the convex area the areas of the two bases.

To find the slant height of a frustum of a right pyramid or right cone, take the difference between the radii of the inscribed circles of the two bases, or between the radii of the two bases. This difference is one side of a right triangle whose other side is the altitude of the frustum. The hypotenuse of this right triangle is the slant height.

EXAMPLE.—A square tower roof has the dimensions shown in Fig. 31. To find the slate required to cover the tower the convex area is required. What is this convex area?

SOLUTION.—The tower has the form of a frustum of a pyramid, 6 ft. 6 in. square at the bottom and 3 ft. 6 in. square at the top, 6 ft. being the altitude. 6 ft. 6 in. $=6\frac{1}{2}$ ft. and 3 ft. 6 in. $=3\frac{1}{2}$ ft. The perimeter of the lower base is $4\times6\frac{1}{2}=26$ ft., and that of the upper base is $4\times3\frac{1}{2}=14$ ft. The slant height AB of the frustum is the hypotenuse of the right triangle ACB whose height AC is equal to the altitude of the frustum or 6 ft., and whose base BC is equal to the difference between the radii of the circles inscribed in the upper and lower bases, or

$$\tfrac{1}{2}(6\tfrac{1}{2}-3\tfrac{1}{2})=1\tfrac{1}{2}\text{ ft.}$$

The slant height is, therefore,

$$\sqrt{(1\tfrac{1}{2})^2+6^2}=6.18\text{ ft.}$$

Applying the formula,

$$A=\frac{(P+p)s}{2}=\frac{(26+14)\times6.18}{2}=123.6\text{ sq. ft.}\quad\text{Ans.}$$

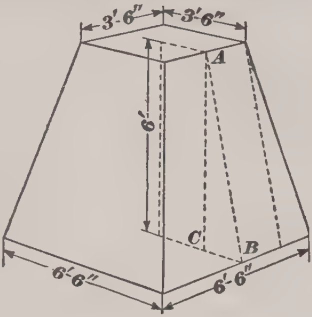


FIG. 31

62. The following rule gives a method of finding the volume of the frustum of a pyramid or cone:

Rule.—*To find the volume of the frustum of a pyramid or cone, add the areas of the upper base, the lower base, and the square root of the product of the areas of the two bases; multiply this sum by one-third of the altitude.*

Let A = area of lower base;
a = area of upper base;
h = altitude;
V = volume.

Then,

$$V=\frac{h}{3}(A+a+\sqrt{A\times a})$$

In this and other formulas of similar form, care should be taken to express the various dimensions in units of like denomination. For example, if the areas are expressed in square inches, the altitude must be in inches, and the result will be in cubic inches; if the areas are in square feet, the altitude must be in feet, and the result will be in cubic feet.

EXAMPLE.—A marble monument is $2\frac{1}{2}$ feet square at the base, 1 foot square at the top, and 16 feet high. If marble weighs 160 pounds per cubic foot, what is the weight of the stone?

SOLUTION.—The area of the lower base is $2\frac{1}{2}\times 2\frac{1}{2}=6.25$ sq. ft; that of the upper base is 1 sq. ft. Apply the formula, and

$$V=\frac{h}{3}(A+a+\sqrt{A\times a})=\frac{16}{3}(6.25+1+\sqrt{6.25\times 1})=52 \text{ cu. ft.}$$

At 160 lb. per cu. ft., the weight is $52\times 160=8{,}320$ lb. Ans.

63. When the bases of the frustum are square, let d and D represent the sides of the bases, respectively, and V the volume of the frustum; then,

$$V=\frac{h}{3}(d^2+d\,D+D^2)$$

When the bases are circular, let d and D represent the respective diameters; then,

$$V=\frac{h}{3}(d^2+d\,D+D^2)\times\frac{\pi}{4}$$

EXAMPLE.—Find the volume of a pail having the form of a frustum of a cone, the upper and lower diameters being 10 and 8 inches, respectively, and the height of pail $9\frac{1}{2}$ inches.

SOLUTION.—Substitute the given values in the formula, and

$$V=\frac{9.5}{3}(8^2+8\times 10+10^2)\times .7854$$

$$=\frac{9.5}{3}\times 244\times .7854=606.85 \text{ cu. in.} \quad \text{Ans.}$$

WEDGE

64. A **wedge,** Fig. 32, is a solid having plane surfaces, of which the base is a parallelogram, the ends are triangles, and the sides are quadrilaterals meeting in a line parallel to the sides of the base.

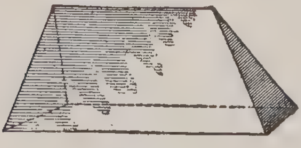


FIG. 32

65. The rule for finding the volume of a wedge is as follows:

Rule.—*To find the volume of a wedge, multiply together the width of the base, the perpendicular distance from the base to the edge, and the sum of the lengths of the three parallel edges; divide the product by 6.*

Let V = volume;
w = width of base;
h = perpendicular distance from base to edge;
s = sum of lengths of the three parallel edges.

Then,
$$V=\frac{w\,h\,s}{6}$$

If the base is a rectangle, and the triangular ends are parallel, the wedge becomes a triangular prism, and the rule for prisms may be used.

Example.—Steel weighs .28 pound per cubic inch. How heavy is a steel wedge 8 inches long, and $1\frac{1}{2}$ in. × 3 in. at the head?

Solution.—Here, w is $1\frac{1}{2}$ in., h is 8 in., s is $3+3+3=9$ in. Then,
$$V=\frac{1.5\times8\times9}{6}=18 \text{ cu. in.}$$

Or, as this wedge is a prism, the volume = area of end × length of edge; area of triangle is
$$\frac{1.5\times8}{2}=6 \text{ sq. in.}$$

Multiply this by the length of the edge, 3 in., and the volume is 18 cu. in. The weight at .28 lb. per cu. in. is
$$.28\times18=5.04 \text{ lb.}\quad \text{Ans.}$$

SPHERE

66. A **sphere,** Fig. 33, is a solid bounded by a uniformly curved surface, every point of which is equally distant from a point within called the center. The word *ball* is commonly used instead of *sphere*.

67. The following rule states the method of finding the area of the surface of a sphere:

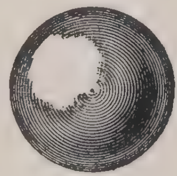


Fig. 33

Rule.—*To find the area of the surface of a sphere, multiply the square of the diameter by 3.1416.*

Let S = surface;
d = diameter of sphere.

Then, $S=\pi d^2=3.1416d^2$

Example.—A ball on a flagstaff is 10 inches in diameter and is to be gilded, but 20 square inches is to be deducted for space covered by attachment to the pole. How many books of gold leaf, each containing 25 leaves $3\frac{3}{8}$ inches square, will be required for gilding it?

Solution.—Apply the formula. $S=\pi\, d^2=3.1416\times 10\times 10=314.16$ sq. in. Deducting 20 sq. in., the net surface is 294.16 sq. in. Each gold leaf has an area of $3\frac{3}{8}$ in. $\times 3\frac{3}{8}$ in., or 11.4 sq. in., nearly; hence, for 294.16 sq. in., there will be needed $294.16\div 11.4=25.8$ leaves, or 1 book and 1 leaf. Ans.

68. The volume of a sphere may be found when the diameter is known by applying the following rule:

Rule.—*To find the volume of a sphere, multiply the cube of the diameter by .5236.*

Let $V=$ volume;
$d=$ diameter.

Then,
$$V=\frac{\pi}{6}d^3=.5236d^3$$

Since $\frac{\pi d^3}{6}=\frac{1}{6}\times\pi d^2\times d$, the volume of a sphere is equal to one-sixth of the surface multiplied by the diameter.

Example.—What is the weight of a cast-iron ball 6 inches in diameter, if the metal weighs .26 pound per cubic inch?

Solution.—Apply the formula, and
$$V=.5236\ d^3=.5236\times 216=113.1 \text{ cu. in.}$$
The weight is therefore $113.1\times .26=29.41$ lb. Ans.

69. The volume of a spherical shell is equal to the difference in volume between two spheres having the outer and inner diameters of the shell.

PRISMOIDS

70. Definition.—A *prismoid* is a solid of which two faces, called bases, are parallel polygons and the other faces, or sides, are triangles, trapezoids, or parallelograms joining the bases. In the prismoid in Fig. 34, the parallel faces *ABCD* and *EFGHI* are the bases, the side *BCGF* is a rectangle, the sides *ABFE*, *ADIE*, and *CDHG* are trapezoids, and the side *DHI* is a triangle. It must be kept in mind that all faces of a prismoid are plane surfaces and each face has at least one vertex in each base. Prisms, pyramids, and frustums of pyramids are special kinds of prismoids.

The altitude of a prismoid is the perpendicular distance between the bases.

71. Middle Section of Prismoid.—If a prismoid is imagined to be cut by a plane parallel to the two bases and midway between them, the intersection is a polygon called the middle section of the prismoid. In Fig. 34, the middle section is the polygon *JKLMN*.

Each vertex of a middle section is the middle point of an edge of the prismoid, and the length of each side of a middle section is equal to one-half the sum of the lengths of the corresponding sides of the bases. Thus, in Fig. 34, *J* is the middle point of *AE*, *K* is the middle point of *BF*, *M* is the middle point of *DH*, etc. Also, $JK=\frac{AB+EF}{2}$, $MN=\frac{HI+0}{2}=\frac{HI}{2}$ etc.

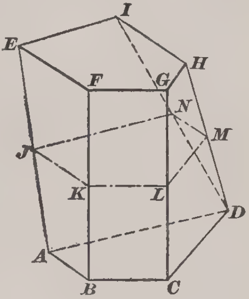


FIG. 34

Except in the case of a prism, where the bases and the middle section are equal, the area of the middle section of a prismoid is not equal to one-half the sum of the areas of the bases. The area of a middle section generally must be computed from its dimensions.

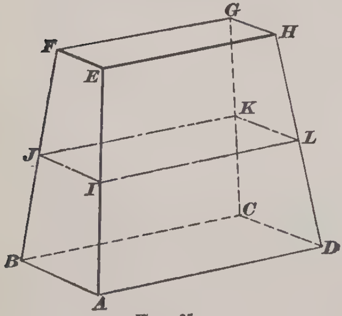


FIG. 35

EXAMPLE.—In Fig. 35 is shown a masonry pier in the shape of a prismoid. The bottom *ABCD* is a rectangle 11 ft.×30 ft. and the top *EFGH* is a rectangle 7 ft.×24 ft. Find the area of the middle section *IJKL*.

SOLUTION.—The middle section is similar to the bases. It is, therefore, a rectangle and its area is equal to the product of *IL* and *IJ*. Since each dimension of the middle section is equal to one-half the sum of the corresponding dimensions of the bases, $IL=\frac{AD+EH}{2}=\frac{30+24}{2}=27$ ft. and $IJ=\frac{AB+EF}{2}=\frac{11+7}{2}=9$ ft. Hence, the area of $IJKL$ equals $27\times9=243$ sq. ft. Ans.

72. Volume of Prismoid.—The volume of a prismoid is equal to one-sixth of the altitude multiplied by the sum of the area of the lower base, the area of the upper base, and four times the area of the middle section.

Let V = volume of prismoid;
h = altitude;
A = area of lower base;
a = area of upper base;
M = area of middle section.

Then,

$$V=\frac{h}{6}(A+a+4M)$$

This formula, which is called the *prismoidal formula,* is applied extensively in practice.

Example.—If the pier in the example of Art. **71** is 30 feet high, how many cubic yards of masonry does it contain?

Solution.—The area A of the lower base of the prismoid equals $30\times11=330$ sq. ft.; the area a of the upper base equals $24\times7=168$ sq. ft.; and the area M of the middle section is 243 sq. ft. By the prismoidal formula,

$$V=\frac{h}{6}(A+a+4M)=\frac{30}{6}(330+168+4\times243)=7{,}350 \text{ cu. ft.}$$

This is equivalent to $\frac{7{,}350}{27}$ or 272.2 cu. yd. Ans.

EXAMPLES FOR PRACTICE

1. A right cylinder is 26 inches in diameter and 32 inches high. Find its convex surface. Ans. 18 sq. ft., approximately

2. On a piece of level ground a cellar 24 feet wide, 36 feet long, and 7 feet deep is dug. How many cubic yards of earth are removed? Ans. 224 cu. yd.

3. A steel shaft is 4 inches in diameter and 10 feet long. What is its weight, if steel weighs .283 pound per cubic inch? Ans. 427 lb., nearly

4. Find the number of board feet in a timber 16 feet long, 12 inches wide, and 10 inches thick. Ans. 160 ft. B. M.

5. A conical tower 8 feet in diameter at the base and 12 feet high is to be painted. Find the area of convex surface. Ans. 159 sq. ft., nearly

6. A block of marble has the form of a pyramid whose base is 2 feet 6 inches square and whose height is 6 feet. If marble weighs 160 pounds per cubic foot, what is the weight of the block? Ans. 1 ton

7. A tank having the shape of a frustum of a cone is 8 feet high. The inside diameters at top and bottom are 5 feet and 6 feet, respectively. (*a*) How many gallons will the tank hold? (*b*) What is the area of the inside convex surface?

Ans. { (*a*) 1,426 gal., nearly
(*b*) 138.5 sq. ft.

8. A spherical balloon is 20 feet in diameter. (*a*) How many cubic feet of gas does it contain? (*b*) How many square feet of surface has it?

Ans. { (*a*) 4,189 cu. ft.
(*b*) 1,257 sq. ft.

SYMMETRICAL AND SIMILAR FIGURES

73. An **axis of symmetry** is any line so drawn that, if the part of the figure on one side of the line is folded over on this line, it will coincide exactly with the other part, point for point and line for line. Thus, in Fig. 36, if the upper semicircle is folded over on the diameter *C D*, it will coincide exactly with the lower semicircle; also, if the part on the right of the diameter *A B* is folded over on *A B*, it will coincide exactly with the part on the left of this line.

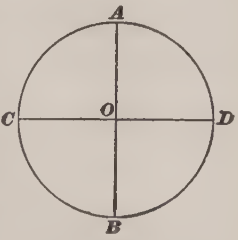


FIG. 36

It is evident from what has just been stated that a circle may have any number of axes of symmetry. In certain cases, however, a figure may be symmetrical with regard to only one axis. Thus, the isosceles triangle *A B C*, Fig. 37, is symmetrical with regard to the axis *B D*, because the part *B C D* would coincide with the part *B A D* if folded over on the line *B D*; but no other axis of symmetry could be drawn. A rectangle has two axes of symmetry at right angles to each other. A regular hexagon has six axes of symmetry.

74. **Similar figures** are those that are alike in form. As in the case of polygons, which have been considered, two figures, to be simliar, must have their corresponding sides in proportion, and the angles of one equal to the corresponding angles of the other. Circles are, of course, similar figures.

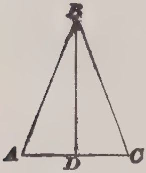


FIG. 37

75. *The areas of two similar figures are to each other as the squares of any one dimension.* Thus, a regular octagon whose sides are 1 inch long contains 4.828 square inches; another with sides 4 inches long contains 4^2, or 16, times 4.828 square inches $=77.25$ square inches; for, let $A=$ required area, then $A:4.828$ sq. in. $=4^2:1^2$, or $A=16\times4.828$ square inches.

The use of this principle often saves considerable labor in determining the areas of similar figures, as, for instance, the end areas of frustums of pyramids, cones, etc.

76. The volumes and weights of similar solids are to each other as the cubes of any one dimension.

EXAMPLE.—If a cast-iron ball 9 inches in diameter weighs 100 pounds, what would a ball 15 inches in diameter weigh?

SOLUTION.— $100:x=9^3:15^3$, or

$$x=\frac{100\times3{,}375}{729}=462.96 \text{ lb.}$$

The weight of the larger ball is, therefore, 462.96 lb. Ans.

APPENDIX

PARABOLA

77. Properties of Parabola.—A parabola, Fig. 38, is a plane curve so formed that any point P on it is equally distant from a fixed point F and a fixed line DE; that is, $PF = PN$. The fixed point F is called the *focus*, and the fixed line DE is the *directrix*. The line AX that passes through the focus and is perpendicular to the directrix is the *axis* of the curve, and the point O where the curve crosses the axis is the *vertex*. Any line, as PP', which is perpendicular to the axis and is terminated by the curve, is called a *double ordinate*. A parabola may be continued indefinitely, but a double ordinate, as QQ', is generally considered to limit the curve. Such a figure, as $QOQ'X$, is often called a *segment of a parabola.*

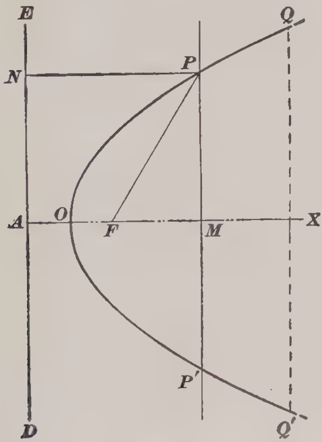


FIG. 38

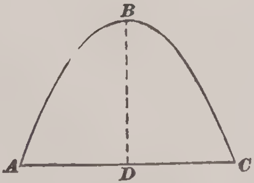


FIG. 39

The axis of a parabola is the perpendicular bisector of each double ordinate; that is, $PM = MP'$ and $QX = XQ'$. The vertex is midway between the focus and the directrix; that is, $AO = OF$.

78. Area of Segment of Parabola.—The area of a segment of a parabola is equal to two-thirds of the product of the length of the bounding double ordinate and the distance from the vertex to that double ordinate.

Let A = area of segment of parabola;
o = length of bounding double ordinate;
m = distance from vertex to double ordinate.

Then, $$A = \tfrac{2}{3}\, om$$

In Fig. 39, the curve ABC is a parabola, the axis BD being vertical and the double ordinate being horizontal. The area of the segment ABC is equal to $\tfrac{2}{3}\, AC \times BD$.